

---

# **Clifford Documentation**

***Release 1.4.0dev0***

**Clifford Team**

**Mar 11, 2021**



# CONTENTS

<b>1</b>	<b>Installation</b>	<b>3</b>
1.1	conda . . . . .	3
1.2	PyPI . . . . .	3
<b>2</b>	<b>API</b>	<b>5</b>
2.1	clifford (clifford) . . . . .	5
2.1.1	Constructing algebras . . . . .	5
2.1.2	Global configuration functions . . . . .	16
2.1.3	Miscellaneous classes . . . . .	16
2.1.4	Miscellaneous functions . . . . .	18
2.2	cga (clifford.cga) . . . . .	18
2.2.1	The CGA . . . . .	19
2.2.2	Objects . . . . .	21
2.2.3	Operators . . . . .	25
2.2.4	Meta-Class . . . . .	29
2.3	tools (clifford.tools) . . . . .	30
2.3.1	Tools for specific ga's . . . . .	30
2.3.2	Classifying conformal GAs . . . . .	57
2.3.3	Determining Rotors From Frame Pairs or Orthogonal Matrices . . . . .	60
2.4	operator functions (clifford.operator) . . . . .	63
2.5	transformations (clifford.transformations) . . . . .	63
2.5.1	Base classes . . . . .	64
2.5.2	Matrix-backed implementations . . . . .	64
2.5.3	Helper functions . . . . .	66
<b>3</b>	<b>Predefined Algebras</b>	<b>69</b>
<b>4</b>	<b>Changelog</b>	<b>71</b>
4.1	Changes in 1.4.x . . . . .	71
4.2	Changes in 1.3.x . . . . .	71
4.2.1	Bugs fixed . . . . .	72
4.2.2	Compatibility notes . . . . .	72
4.2.3	Patch releases . . . . .	73
4.3	Changes in 1.2.x . . . . .	73
4.3.1	Bugs fixed . . . . .	73
4.3.2	Compatibility notes . . . . .	73
4.4	Changes in 1.1.x . . . . .	73
4.4.1	Compatibility notes . . . . .	74
4.4.2	Bugs fixed . . . . .	74
4.4.3	Internal changes . . . . .	74

4.5	Changes 0.6-0.7 . . . . .	74
4.6	Changes 0.5-0.6 . . . . .	74
4.7	Acknowledgements . . . . .	75
<b>5</b>	<b>Issues</b>	<b>77</b>
<b>6</b>	<b>Quick Start (G2)</b>	<b>79</b>
6.1	Setup . . . . .	79
6.2	Basics . . . . .	79
6.3	Reflection . . . . .	80
6.4	Rotation . . . . .	80
<b>7</b>	<b>The Algebra Of Space (G3)</b>	<b>81</b>
7.1	Setup . . . . .	81
7.2	Basics . . . . .	82
7.2.1	Products . . . . .	82
7.2.2	Defects in Precedence . . . . .	82
7.2.3	Multivectors . . . . .	83
7.2.4	Reversion . . . . .	83
7.2.5	Grade Projection . . . . .	83
7.2.6	Magnitude . . . . .	84
7.2.7	Inverse . . . . .	84
7.2.8	Dual . . . . .	84
7.2.9	Pretty, Ugly, and Display Precision . . . . .	84
7.3	Applications . . . . .	85
7.3.1	Reflections . . . . .	85
7.3.2	Rotations . . . . .	85
7.3.3	Some Ways to use Functions . . . . .	86
7.4	Changing Basis Names . . . . .	88
<b>8</b>	<b>Rotations in Space: Euler Angles, Matrices, and Quaternions</b>	<b>89</b>
8.1	Euler Angles with Rotors . . . . .	89
8.2	Implementation of Euler Angles . . . . .	90
8.3	Convert to Quaternions . . . . .	90
8.4	Convert to Rotation Matrix . . . . .	91
8.5	Convert a Rotation Matrix to a Rotor . . . . .	91
<b>9</b>	<b>Space Time Algebra</b>	<b>93</b>
9.1	Intro . . . . .	93
9.2	Setup . . . . .	93
9.3	The Space Time Split . . . . .	93
9.4	Splitting a space-time vector (an event) . . . . .	94
9.5	Splitting a Bivector . . . . .	95
9.6	Lorentz Transformations . . . . .	95
9.7	Lorentz Invariants . . . . .	96
<b>10</b>	<b>Interfacing Other Mathematical Systems</b>	<b>97</b>
10.1	Complex Numbers . . . . .	97
10.2	Quaternions . . . . .	99
<b>11</b>	<b>Writing high(ish) performance code with Clifford and Numba via Numpy</b>	<b>103</b>
11.1	First import the Clifford library as well as numpy and numba . . . . .	103
11.2	Choose a specific space . . . . .	103
11.3	Performance of mathematically idiomatic Clifford algorithms . . . . .	104
11.4	Canonical blade coefficient representation in Clifford . . . . .	105

11.5	Exploiting blade representation to write a fast function . . . . .	105
11.6	Composing larger functions . . . . .	107
11.7	Algorithms exploiting known sparseness of MultiVector value array . . . . .	108
11.8	Future work on performance . . . . .	111
<b>12</b>	<b>Conformal Geometric Algebra</b>	<b>113</b>
12.1	Using <code>conformalize()</code> . . . . .	113
12.2	Operations . . . . .	115
12.2.1	Versors purely in $E_0$ . . . . .	115
12.2.2	Versors partly in $E_0$ . . . . .	116
12.2.3	Versors Out of $E_0$ . . . . .	117
12.2.4	Combinations of Operations . . . . .	117
12.3	More examples . . . . .	118
12.3.1	Visualization tools . . . . .	118
12.3.2	Object Oriented CGA . . . . .	123
12.3.3	Example 1 Interpolating Conformal Objects . . . . .	126
12.3.4	Example 2 Clustering Geometric Objects . . . . .	130
12.3.5	Application to Robotic Manipulators . . . . .	133
<b>13</b>	<b>Linear transformations</b>	<b>141</b>
13.1	Vector transformations in linear algebra . . . . .	141
13.2	Multivector transformations in geometric algebra . . . . .	142
13.2.1	Matrix representation . . . . .	143
<b>14</b>	<b>Matrix Representations of Geometric Functions</b>	<b>145</b>
14.1	Anti-symmetric . . . . .	146
14.2	Diagonal ( Directional Scaling) . . . . .	146
14.3	Orthogonal, Rotation . . . . .	146
14.4	Orthogonal, Reflection . . . . .	147
14.5	Symmetric . . . . .	148
14.6	Eigen stuffs . . . . .	148
14.6.1	Rotation . . . . .	148
14.6.2	Symmetric . . . . .	149
14.7	Primitive Visualization in 2D . . . . .	149
<b>15</b>	<b>Working with custom algebras</b>	<b>151</b>
15.1	Visualization . . . . .	152
15.2	Apollonius' problem in $\mathbb{R}^2$ with circles . . . . .	153
15.3	Apollonius' problem in $\mathbb{R}^3$ with spheres . . . . .	154
<b>16</b>	<b>Other resources for <code>clifford</code></b>	<b>155</b>
16.1	Slide Decks . . . . .	155
16.2	Videos . . . . .	155
<b>17</b>	<b>Other resources for Geometric Algebra</b>	<b>157</b>
17.1	Links . . . . .	157
17.2	Introductory textbooks . . . . .	157
<b>18</b>	<b>Bibliography</b>	<b>159</b>
	<b>Bibliography</b>	<b>161</b>
	<b>Python Module Index</b>	<b>163</b>
	<b>Index</b>	<b>165</b>



```

In [1]: from clifford.g3 import * # import GA for 3D space

In [2]: import math

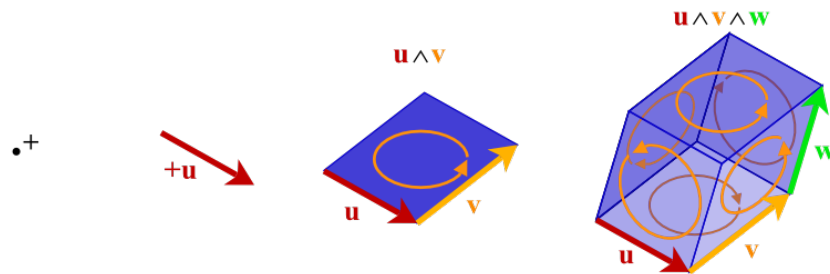
In [3]: a = e1 + 2*e2 + 3*e3 # vector

In [4]: R = math.exp(math.pi/4*e12) # rotor

In [5]: R*a~R # rotate the vector
Out[5]: (2.0^e1) - (1.0^e2) + (3.0^e3)

```

This module implements Geometric Algebras (a.k.a. Clifford algebras). Geometric Algebra (GA) is a universal algebra which subsumes complex algebra, quaternions, linear algebra and several other independent mathematical systems. Scalars, vectors, and higher-grade entities can be mixed freely and consistently in the form of mixed-grade multivectors.







## INSTALLATION

Clifford can be installed via the standard mechanisms for Python packages, and is available through both PyPI and Anaconda.

### 1.1 conda

If you are using the [Anaconda python distribution](#), then you should install using the `conda` command.

Once you have anaconda installed you can install clifford from the conda-forge channel with the command:

```
conda install -c conda-forge clifford
```

### 1.2 PyPI

If you are not using Anaconda, you can install `clifford` from PyPI using the `pip` command that comes with Python:

```
pip3 install clifford
```

**Note:** The 1.3 release and older are not compatible with `numba 0.50.0`, and emit a large number of warnings on 0.49.0. To avoid these warnings, install with:

```
pip3 install clifford "numba~=0.48.0" "sparse~=0.9.0"
```

If you want the development version of `clifford` you can `pip install` from the latest master with:

```
pip3 install git+https://github.com/pygae/clifford.git@master
```

If you want to modify `clifford` itself, then you should use an editable (`-e`) installation:

```
git clone https://github.com/pygae/clifford.git
pip3 install -e ./clifford
```

If you are not running as `sudo`, and your python installation is system-wide, you will need to pass `--user` to the invocations above.



## 2.1 clifford (clifford)

The top-level module. Provides two core classes, *Layout* and *MultiVector*, along with several helper functions to implement the algebras.

### 2.1.1 Constructing algebras

Note that typically the predefined-algebras are sufficient, and there is no need to build an algebra from scratch.

<code>Cl([p, q, r, sig, names, firstIdx, mvClass])</code>	Returns a <i>Layout</i> and basis blade <i>MultiVectors</i> for the geometric algebra $Cl_{p,q,r}$ .
<code>conformalize(layout[, added_sig, mvClass])</code>	Conformalize a Geometric Algebra

#### clifford.Cl

`clifford.Cl(p=0, q=0, r=0, sig=None, names=None, firstIdx=1, mvClass=<class 'clifford._multivector.MultiVector'>)`

Returns a *Layout* and basis blade *MultiVectors* for the geometric algebra  $Cl_{p,q,r}$ .

The notation  $Cl_{p,q,r}$  means that the algebra is  $p + q + r$ -dimensional, with the first  $p$  vectors with positive signature, the next  $q$  vectors negative, and the final  $r$  vectors with null signature.

#### Parameters

- **p** (*int*) – number of positive signature basis vectors
- **q** (*int*) – number of negative signature basis vectors
- **r** (*int*) – number of zero signature basis vectors
- **sig** – See the docs for *clifford.Layout*. If *sig* is passed, then *p*, *q*, and *r* are ignored.
- **names** – See the docs for *clifford.Layout*. If *sig* is passed, then *p*, *q*, and *r* are ignored.
- **firstIdx** – See the docs for *clifford.Layout*. If *sig* is passed, then *p*, *q*, and *r* are ignored.

#### Returns

- **layout** (*Layout*) – The resulting layout
- **blades** (*Dict[str, MultiVector]*) – The blades of the returned layout, equivalent to `layout.blades`.

## clifford.conformalize

```
clifford.conformalize(layout, added_sig=[1, -1], *, mvClass=<class 'clifford._multivector.MultiVector'>, **kwargs)
```

Conformalize a Geometric Algebra

Given the *Layout* for a GA of signature (p, q), this will produce a GA of signature (p+1, q+1), as well as return a new list of blades and some *stuff*. *stuff* is a dict containing the null basis blades, and some up/down functions for projecting in/out of the CGA.

### Parameters

- **layout** (*clifford.Layout*) – layout of the GA to conformalize (the base)
- **added\_sig** (*list-like*) – list of +1, -1 denoted the added signatures
- **\*\*kwargs** – passed to `Cl()` used to generate conformal layout

### Returns

- **layout\_c** (*ConformalLayout*) – layout of the base GA
- **blades\_c** (*dict*) – blades for the CGA
- **stuff** (*dict*) – dict mapping the following members of *ConformalLayout* by their names, for easy unpacking into the global namespace:

<i>up</i> (x)	up-project a vector from GA to CGA
<i>down</i> (x)	down-project a vector from CGA to GA
<i>homo</i> (x)	homogenize a CGA vector

## Examples

```
>>> from clifford import Cl, conformalize
>>> G2, blades = Cl(2)
>>> G2C, bladesC, stuff = conformalize(G2)
>>> locals().update(bladesC)
>>> locals().update(stuff)
```

Whether you construct your algebras from scratch, or use the predefined ones, you'll end up working with the following types:

<i>MultiVector</i> (layout[, value, string])	An element of the algebra
<i>Layout</i> (sig, *, ids, order, names)	Layout stores information regarding the geometric algebra itself and the internal representation of multivectors.
<i>ConformalLayout</i> (*args[, layout])	A layout for a conformal algebra, which adds extra constants and helpers.

**clifford.MultiVector**

**class** clifford.**MultiVector** (*layout*, *value=None*, *string=None*, \*, *dtype: numpy.dtype = <class 'numpy.float64'>*)

An element of the algebra

**Parameters**

- **layout** (instance of `clifford.Layout`) – The layout of the algebra
- **value** (sequence of length `layout.gaDims`) – The coefficients of the base blades
- **dtype** (`numpy.dtype`) – The datatype to use for the multivector, if no value was passed.

New in version 1.1.0.

**Notes**

The following operators are overloaded:

- $A * B$  : geometric product
- $A \wedge B$  : outer product
- $A \mid B$  : inner product
- $A \ll B$  : left contraction
- $\sim M$  : reversion
- $M(N)$  : grade or subspace projection
- $M[N]$  : blade projection

**exp** () → clifford.\_multivector.MultiVector

**vee** (*other*) → clifford.\_multivector.MultiVector  
Vee product  $A \vee B$ .

This is often defined as:

$$(A \vee B)^* = A^* \wedge B^* \\ \implies A \vee B = (A^* \wedge B^*)^{-*}$$

This is very similar to the `meet()` function, but always uses the dual in the full space .

Internally, this is actually implemented using the complement functions instead, as these work in degenerate metrics like PGA too, and are equivalent but faster in other metrics.

**\_\_and\_\_** (*other*) → clifford.\_multivector.MultiVector  
self & other, an alias for `vee()`

**\_\_mul\_\_** (*other*) → clifford.\_multivector.MultiVector  
self \* other, the geometric product  $MN$

**\_\_xor\_\_** (*other*) → clifford.\_multivector.MultiVector  
self ^ other, the Outer product  $M \wedge N$

**\_\_or\_\_** (*other*) → clifford.\_multivector.MultiVector  
self | other, the inner product  $M \cdot N$

**\_\_add\_\_** (*other*) → clifford.\_multivector.MultiVector  
self + other, addition

**\_\_sub\_\_** (*other*) → clifford.\_multivector.MultiVector  
self - other, Subtraction

**right\_complement** () → clifford.\_multivector.MultiVector

**left\_complement** () → clifford.\_multivector.MultiVector

**as\_array** () → [numpy.ndarray](#)

**mag2** () → [numbers.Number](#)  
Magnitude (modulus) squared,  $|M|^2$   
Note in mixed signature spaces this may be negative

**adjoint** () → clifford.\_multivector.MultiVector  
Adjoint / reversion,  $\tilde{M}$   
Aliased as  $\sim M$  to reflect  $\tilde{M}$ , one of several conflicting notations.  
Note that  $\sim (N * M) == \sim M * \sim N$ .

**\_\_invert\_\_** () → clifford.\_multivector.MultiVector  
Adjoint / reversion,  $\tilde{M}$   
Aliased as  $\sim M$  to reflect  $\tilde{M}$ , one of several conflicting notations.  
Note that  $\sim (N * M) == \sim M * \sim N$ .

**\_\_getitem\_\_** (*key: Union[clifford.\_multivector.MultiVector, tuple, int]*) → [numbers.Number](#)  
value = self[key].  
If key is a blade tuple (e.g. (0, 1) or (1, 3)), or a blade, (e.g. e12), then return the (real) value of that blade's coefficient.  
Deprecated since version 1.4.0: If an integer is passed, it is treated as an index into self.value. Use self.value[i] directly.

**\_\_call\_\_** (*other, \*others*) → clifford.\_multivector.MultiVector  
Return a new multi-vector projected onto a grade or another MultiVector  
 $M(g_1, \dots g_n)$  gives  $\langle M \rangle_{g_1} + \dots + \langle M \rangle_{g_n}$   
 $M(N)$  calls [project\(\)](#) as  $N.project(M)$ .

## Examples

```
>>> from clifford.g2 import *
>>> M = 1 + 2*e1 + 3*e12
>>> M(0)
1
>>> M(0, 2)
1 + (3^e12)
```

**clean** (*eps=None*) → clifford.\_multivector.MultiVector  
Sets coefficients whose absolute value is < eps to exactly 0.  
eps defaults to the current value of the global \_settings.\_eps.

**round** (*eps=None*) → clifford.\_multivector.MultiVector  
Rounds all coefficients according to Python's rounding rules.  
eps defaults to the current value of the global \_settings.\_eps.

**lc** (*other*) → clifford.\_multivector.MultiVector

The left-contraction of two multivectors,  $M \rfloor N$

**property pseudoScalar**

Returns a MultiVector that is the pseudoscalar of this space.

**property I**

Returns a MultiVector that is the pseudoscalar of this space.

**invPS** () → clifford.\_multivector.MultiVector

Returns the inverse of the pseudoscalar of the algebra.

**isScalar** () → bool

Returns true iff self is a scalar.

**isBlade** () → bool

Returns true if multivector is a blade.

**isVersor** () → bool

Returns true if multivector is a versor. From [DFM09] section 21.5, definition from 7.6.4

**grades** (*eps=None*) → Set[int]

Return the grades contained in the multivector.

Changed in version 1.1.0: Now returns a set instead of a list

Changed in version 1.3.0: Accepts an *eps* argument

**property blades\_list**

ordered list of blades present in this MV

**normal** () → clifford.\_multivector.MultiVector

Return the (mostly) normalized multivector.

The *\_mostly\_* comes from the fact that some multivectors have a negative squared-magnitude. So, without introducing formally imaginary numbers, we can only fix the normalized multivector's magnitude to  $\pm 1$ .

$\frac{M}{|M|}$  up to a sign

**hitzer\_inverse** ()

**leftLaInv** () → clifford.\_multivector.MultiVector

Return left-inverse using a computational linear algebra method proposed by Christian Perwass.

**normalInv** (*check=True*) → clifford.\_multivector.MultiVector

The inverse of itself if  $M\tilde{M} = |M|^2$ .

$$M^{-1} = \tilde{M}/(M\tilde{M})$$

**Parameters** **check** (*bool*) – When true, the default, validate that it is appropriate to use this method of inversion.

**inv** () → clifford.\_multivector.MultiVector

**leftInv** () → clifford.\_multivector.MultiVector

Return left-inverse using a computational linear algebra method proposed by Christian Perwass.

**rightInv** () → clifford.\_multivector.MultiVector

Return left-inverse using a computational linear algebra method proposed by Christian Perwass.

**dual** (*I=None*) → clifford.\_multivector.MultiVector

The dual of the multivector against the given subspace  $I$ ,  $\tilde{M} = MI^{-1}$

$I$  defaults to the pseudoscalar.

**commutator** (*other*) → clifford.\_multivector.MultiVector

The commutator product of two multivectors.

$$[M, N] = M \times N = (MN + NM)/2$$

**x** (*other*) → clifford.\_multivector.MultiVector

The commutator product of two multivectors.

$$[M, N] = M \times N = (MN + NM)/2$$

**anticommutator** (*other*) → clifford.\_multivector.MultiVector

The anti-commutator product of two multivectors,  $(MN - NM)/2$

**gradeInvol** () → clifford.\_multivector.MultiVector

The grade involution of the multivector.

$$M^* = \sum_{i=0}^{\text{dims}} (-1)^i \langle M \rangle_i$$

**property even**

Even part of this multivector

defined as  $M + M.\text{gradInvol}()$

**property odd**

Odd part of this multivector

defined as  $M - M.\text{gradInvol}()$

**conjugate** () → clifford.\_multivector.MultiVector

The Clifford conjugate (reversion and grade involution).

$$M^* = (\sim M).\text{gradeInvol}()$$

**randomMV** (\*args, \*\*kw)

create a random MV within space of self

This is a wrapper for *randomMV*

**See also:**

[Layout.randomMV](#), [randomMV](#)

**randomV** (\*args, \*\*kw)

create a random MV within space of self

This is a wrapper for [Layout.randomMV](#)

**See also:**

[Layout.randomMV](#)

**project** (*other*) → clifford.\_multivector.MultiVector

Projects the multivector onto the subspace represented by this blade.

$$P_A(M) = (M \rfloor A) A^{-1}$$

**factorise** () → Tuple[List[clifford.\_multivector.MultiVector], [numbers.Number](#)]

Factorises a blade into basis vectors and an overall scale.

Uses the algorithm from [DFM09], section 21.6.

**basis** () → List[clifford.\_multivector.MultiVector]

Finds a vector basis of this subspace.



**join** (*other*) → clifford.\_multivector.MultiVector

The join of two blades,  $J = A \cup B$

Similar to the wedge,  $W = A \wedge B$ , but without decaying to 0 for blades which share a vector.

**meet** (*other*, *subspace=None*) → clifford.\_multivector.MultiVector

The meet of two blades,  $A \cap B$ .

Computation is done with respect to a subspace that defaults to the `join()` if none is given.

Similar to the `vee()`,  $V = A \vee B$ , but without decaying to 0 for blades lying in the same subspace.

**astype** (*\*args*, *\*\*kwargs*)

Change the underlying scalar type of this vector.

Can be used to force lower-precision floats or integers

See `np.ndarray.astype` for argument descriptions.

## clifford.Layout

**class** clifford.**Layout** (*sig*, \*, *ids=None*, *order=None*, *names=None*)

Layout stores information regarding the geometric algebra itself and the internal representation of multivectors.

### Parameters

- **sig** (*List[int]*) – The signature of the vector space. This should be a list of positive and negative numbers where the sign determines the sign of the inner product of the corresponding vector with itself. The values are irrelevant except for sign. This list also determines the dimensionality of the vectors.

Examples:

```
sig=[+1, -1, -1, -1] # Hestenes', et al. Space-Time Algebra
sig=[+1, +1, +1]     # 3-D Euclidean signature
```

- **ids** (*Optional[BasisVectorIds[Any]]*) – A list of ids to associate with each basis vector. These ids are used to generate names (if not passed explicitly), and also used when using tuple-notation to access elements, such as `mv[(1, 3)] = 1`. Defaults to `BasisVectorIds.ordered_integers(len(sig))`; that is, integers starting at 1. This supersedes the old `firstIdx` argument.

Examples:

```
ids=BasisVectorIds.ordered_integers(2, first_index=1)
ids=BasisVectorIds([10, 20, 30])
```

New in version 1.3.0.

- **order** (*Optional[BasisBladeOrder]*) – A specification of the memory order to use when storing the basis blades. Defaults to `BasisBladeOrder.shortlex(len(sig))`. This supersedes the old `bladeTupList` argument.

**Warning:** Various tools within clifford assume this default, so do not change this unless you know what you're doing!

New in version 1.3.0.

- **bladeTupList** (*List[Tuple[int, ...]]*) – List of tuples corresponding to the blades in the whole algebra. This list determines the order of coefficients in the internal representation of multivectors. The entry for the scalar must be an empty tuple, and the entries for grade-1 vectors must be singleton tuples. Remember, the length of the list will be `2**dims`.

Example:

```
bladeTupList = [(), (0,), (1,), (0, 1)] # 2-D
```

Deprecated since version 1.3.0: Use the new *order* and *ids* arguments instead. The above example can be spelt with the slightly longer:

```
ids = BasisVectorIds([.ordered_integers(2, first_index=0)])
order = ids.order_from_tuples([(), (0,), (1,), (0, 1)])
Layout(sig, ids=ids, order=order)
```

- **firstIdx** (*int*) – The index of the first vector. That is, some systems number the base vectors starting with 0, some with 1.

Deprecated since version 1.3.0: Use the new *ids* argument instead, for which the docs show an equivalent replacement

- **names** (*List[str]*) – List of names of each blade. When pretty-printing multivectors, use these symbols for the blades. names should be in the same order as *order*. You may use an empty string for scalars. By default, the name for each non-scalar blade is 'e' plus the ids of the blade as given in *ids*.

Example:

```
names=['', 's0', 's1', 'i'] # 2-D
```

**dims**

dimensionality of vectors (`len(self.sig)`)

**sig**

normalized signature, with all values +1 or -1

**bladeTupList**

list of blades

**gaDims**

`2**dims`

**names**

pretty-printing symbols for the blades

**property gradeList****gmt**

Multiplication table for the geometric product.

This is a tensor of rank 3 such that  $a = bc$  can be computed as  $a_j = \sum_{i,k} b_i M_{ijk} c_k$ .

**omt**

Multiplication table for the inner product, stored in the same way as *gmt*

**imt**

Multiplication table for the outer product, stored in the same way as *gmt*

**lcmt**

Multiplication table for the left-contraction, stored in the same way as *gmt*

**bladeTupList**

**property firstIdx**

Starting point for vector indices

Deprecated since version 1.3.0: This attribute has been deprecated, to match the deprecation of the matching argument in the constructor. Internal code should be using `self._basis_vector_ids.values[x]` instead of `x + self.firstIdx`. This replacement API is not yet finalized, so if you need it please file an issue on github!

**dual\_func**

Generates the dual function for the pseudoscalar

**vee\_func**

Generates the vee product function

**parse\_multivector** (*mv\_string: str*) → `clifford._multivector.MultiVector`

Parses a multivector string into a MultiVector object

**gmt\_func\_generator** (*grades\_a=None, grades\_b=None, filter\_mask=None*)

**imt\_func\_generator** (*grades\_a=None, grades\_b=None, filter\_mask=None*)

**omt\_func\_generator** (*grades\_a=None, grades\_b=None, filter\_mask=None*)

**lcmt\_func\_generator** (*grades\_a=None, grades\_b=None, filter\_mask=None*)

**get\_grade\_projection\_matrix** (*grade*)

Returns the matrix  $M_g$  that performs grade projection via left multiplication eg.  $M_g @ A.value = A(g).value$

**gmt\_func**

**imt\_func**

**omt\_func**

**lcmt\_func**

**left\_complement\_func**

**right\_complement\_func**

**adjoint\_func**

This function returns a fast jitted adjoint function

**inv\_func**

Get a function that returns left-inverse using a computational linear algebra method proposed by Christian Perwass.

$$\begin{matrix} & -1 & & -1 \\ & & & \end{matrix}$$

$M$  where  $M * M == 1$

**get\_left\_gmt\_matrix** (*x*)

This produces the matrix  $X$  that performs left multiplication with  $x$  eg.  $X @ b == (x * b).value$

**get\_right\_gmt\_matrix** (*x*)

This produces the matrix  $X$  that performs right multiplication with  $x$  eg.  $X @ b == (b * x).value$

**load\_ga\_file** (*filename: str*) → `clifford._mvarray.MVArray`

Loads the data from a ga file, checking it matches this layout.

**grade\_mask** (*grade: int*) → `numpy.ndarray`

**property rotor\_mask**

**property metric**

**property scalar**

the scalar of value 1, for this GA (a MultiVector object)

useful for forcing a MultiVector type

**property pseudoScalar**

The pseudoscalar,  $I$ .

**property I**

The pseudoscalar,  $I$ .

**randomMV** ( $n=1$ , *\*\*kwargs*)  $\rightarrow$  clifford.\_multivector.MultiVector

Convenience method to create a random multivector.

see *clifford.randomMV* for details

**randomIntMV** ( $n=1$ , *\*\*kwargs*)  $\rightarrow$  clifford.\_multivector.MultiVector

Convenience method to create a random multivector with integer values for components. Nice for reading,

see *clifford.randomIntMV* for details

**randomV** ( $n=1$ , *\*\*kwargs*)  $\rightarrow$  clifford.\_multivector.MultiVector

generate  $n$  random 1-vector s

**randomRotor** ()  $\rightarrow$  clifford.\_multivector.MultiVector

generate a random Rotor.

this is created by multiplying an  $N$  unit vectors, where  $N$  is the dimension of the algebra if its even; else its one less.

**property basis\_vectors**

dictionary of basis vectors

**property basis\_names**

Get the names of the basis vectors, in the order they are stored.

Changed in version 1.3.0: Returns a list instead of a numpy array

**property basis\_vectors\_lst**

Like *blades\_of\_grade(1)*, but ordered based on the *ids* parameter passed at construction.

**blades\_of\_grade** (*grade: int*)  $\rightarrow$  List[clifford.\_multivector.MultiVector]

return all blades of a given grade,

**property blades\_list**

List of blades in this layout matching the *order* argument this layout was constructed from.

**property blades**

**bases** (*mvClass=<class 'clifford.\_multivector.MultiVector'>*, *grades: Optional[Container[int]] = None*)  $\rightarrow$  Dict[str, clifford.\_multivector.MultiVector]

Returns a dictionary mapping basis element names to their MultiVector instances, optionally for specific grades

if you are lazy, you might do this to populate your namespace with the variables of a given layout.

```
>>> locals().update(layout.blades())
```

Changed in version 1.1.0: This dictionary includes the scalar

**MultiVector** (*\*args, \*\*kwargs*)  $\rightarrow$  clifford.\_multivector.MultiVector

Create a multivector in this layout

convenience func to `MultiVector(layout)`

## clifford.ConformalLayout

**class** `clifford.ConformalLayout(*args, layout=None, **kwargs)`

Bases: `clifford._layout.Layout`

A layout for a conformal algebra, which adds extra constants and helpers.

Typically these should be constructed via `clifford.conformalize()`.

New in version 1.2.0.

**ep**

The first added basis element,  $e_+$ , usually with  $e_+^2 = +1$

**Type** *MultiVector*

**en**

The first added basis element,  $e_-$ , usually with  $e_-^2 = -1$

**Type** *MultiVector*

**eo**

The null basis vector at the origin,  $e_o = 0.5(e_- - e_+)$

**Type** *MultiVector*

**einf**

The null vector at infinity,  $e_\infty = e_- + e_+$

**Type** *MultiVector*

**E0**

The minkowski subspace bivector,  $e_\infty \wedge e_o$

**Type** *MultiVector*

**I\_base**

The pseudoscalar of the base ga, in cga layout

**Type** *MultiVector*

**up** ( $x$ : *clifford.\_multivector.MultiVector*)  $\rightarrow$  *clifford.\_multivector.MultiVector*

up-project a vector from GA to CGA

**homo** ( $x$ : *clifford.\_multivector.MultiVector*)  $\rightarrow$  *clifford.\_multivector.MultiVector*

homogenize a CGA vector

**down** ( $x$ : *clifford.\_multivector.MultiVector*)  $\rightarrow$  *clifford.\_multivector.MultiVector*

down-project a vector from CGA to GA

## Advanced algebra configuration

It is unlikely you will need these features, but they remain as a better spelling for features which have always been in `clifford`.

<code>BasisBladeOrder(bitmap)</code>	Represents the storage order in memory of basis blade coefficients.
<code>BasisVectorIds(blade_ids)</code>	Stores ids for the ordered set of basis vectors, typically integers.

### 2.1.2 Global configuration functions

These functions are used to change the global behavior of `clifford`.

`clifford.eps` (*newEps=None*)

Get/Set the epsilon for float comparisons.

`clifford.pretty` (*precision=None*)

Makes `repr` (`MultiVector`) default to pretty-print.

*precision* arg can be used to set the printed precision.

**Parameters** `precision` (*int*) – number of sig figs to print past decimal

#### Examples

```
>>> pretty(5)
```

`clifford.ugly` ()

Makes `repr` (`MultiVector`) default to eval-able representation.

`clifford.print_precision` (*newVal*)

Set the epsilon for float comparisons.

**Parameters** `newVal` (*int*) – number of sig figs to print (see builtin *round*)

#### Examples

```
>>> print_precision(5)
```

### 2.1.3 Miscellaneous classes

<code>MVArray(input_array)</code>	<code>MultiVector</code> Array
<code>Frame(input_array)</code>	A frame of vectors
<code>BladeMap</code> ( <code>blades_map</code> [, <code>map_scalars</code> ])	A Map Relating Blades in two different algebras

**clifford.Frame****class** clifford.**Frame**(*input\_array*)

Bases: clifford.\_mvarray.MVArray

A frame of vectors

**property** **En**Volume element for this frame,  $E_n = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ **property** **inv**

The inverse frame of self

**is\_innarmorphic\_to**(*other: clifford.\_frame.Frame*, *eps: Optional[float] = None*) → boolIs this frame *innarmorphic* to other?*innarmorphic* means both frames share the same inner-product between corresponding vectors. This implies that the two frames are related by an orthogonal transform.**clifford.BladeMap****class** clifford.**BladeMap**(*blades\_map*, *map\_scalars=True*)

A Map Relating Blades in two different algebras

**Examples**

```
>>> from clifford import Cl
>>> # Dirac Algebra `D`
>>> D, D_blades = Cl(1, 3, firstIdx=0, names='d')
>>> locals().update(D_blades)
```

```
>>> # Pauli Algebra `P`
>>> P, P_blades = Cl(3, names='p')
>>> locals().update(P_blades)
>>> sta_split = BladeMap([(d01, p1),
...                       (d02, p2),
...                       (d03, p3),
...                       (d12, p12),
...                       (d23, p23),
...                       (d13, p13)])
```

**property** **b1****property** **b2****property** **layout1****property** **layout2**

## 2.1.4 Miscellaneous functions

<code>grade_obj(objin[, threshold])</code>	Returns the modal grade of a multivector
<code>randomMV(layout[, min, max, grades, ...])</code>	n Random MultiVectors with given layout.
<code>randomIntMV(layout[, min, max, uniform])</code>	Random MultiVectors with given layout.

### `clifford.grade_obj`

`clifford.grade_obj(objin, threshold=1e-07)`  
Returns the modal grade of a multivector

### `clifford.randomMV`

`clifford.randomMV(layout, min=-2.0, max=2.0, grades=None, mvClass=<class 'clifford._multivector.MultiVector'>, uniform=None, n=1, normed=False)`  
n Random MultiVectors with given layout.

Coefficients are between min and max, and if grades is a list of integers, only those grades will be non-zero.

#### Parameters

- **layout** (*Layout*) – the layout
- **min** (*Number*) – minimum of random range
- **max** (*Number*) – maximum
- **grades** (*int*, *list*) – number or list of grades to generate
- **mvClass** (*class*) – the class of MultiVector
- **uniform** (*function*) – a function like `np.random.uniform`, but not limited to
- **n** (*int*) – number of samples to generate
- **normed** (*bool*) – should results be normalized?

#### Examples

```
>>> randomMV(layout, min=-2.0, max=2.0, grades=None, uniform=None, n=2)
```

See also:

`randomIntMV`

## 2.2 `cga (clifford.cga)`

Object Oriented Conformal Geometric Algebra.



## Examples

```
>>> from clifford import Cl
>>> from clifford.cga import CGA
>>> g3, blades = Cl(3)
>>> locals().update(blades)
>>> g3c = CGA(g3)
>>> C = g3c.round(3)           # create random sphere
>>> T = g3c.translation(e1+e2) # create translation
>>> C_ = T(C)                  # translate the sphere
>>> C_.center                  # compute center of sphere
-(1.0^e4) - (1.0^e5)
```

### 2.2.1 The CGA

---

*CGA*(layout\_orig)

---

Conformal Geometric Algebra

---

#### clifford.cga.CGA

**class** clifford.cga.**CGA**(*layout\_orig*)

Conformal Geometric Algebra

conformalizes the *layout\_orig*, and provides several methods and for objects/operators

**Parameters** *layout\_orig* ([*clifford.Layout*, int]) – a layout for the *base* geometric algebra which is conformalized if given as an int, then generates a euclidean space of given dimension

#### Examples

```
>>> from clifford import Cl
>>> from clifford.cga import CGA
>>> g3, blades = Cl(3)
>>> g3c = CGA(g3)
>>> g3c = CGA(3)
```

## Methods

<code>__init__</code>	Initialize self.
<code>base_vector</code>	random vector in the lower(original) space
<code>dilation</code>	see <i>Dilation</i>
<code>flat</code>	see <i>Flat</i>
<code>null_vector</code>	generates random null vector if x is None, or returns a null vector from base vector x, if $x^{\text{self.I\_base}} == 0$ returns x,
<code>rotation</code>	see <i>Rotation</i>
<code>round</code>	see <i>Round</i>
<code>straight_up</code>	place a vector from <i>layout_orig</i> into this CGA, without up()

continues on next page

Table 8 – continued from previous page

---

<i>translation</i>	see <i>Translation</i>
<i>transversion</i>	see <i>Transversion</i>

---

**clifford.cga.CGA.\_\_init\_\_**

**CGA.\_\_init\_\_** (*layout\_orig*) → None  
Initialize self. See help(type(self)) for accurate signature.

**clifford.cga.CGA.base\_vector**

**CGA.base\_vector** () → clifford.\_multivector.MultiVector  
random vector in the lower(original) space

**clifford.cga.CGA.dilation**

**CGA.dilation** (\*args) → clifford.cga.Dilation  
see *Dilation*

**clifford.cga.CGA.flat**

**CGA.flat** (\*args) → clifford.cga.Flat  
see *Flat*

**clifford.cga.CGA.null\_vector**

**CGA.null\_vector** (*x=None*) → clifford.\_multivector.MultiVector  
generates random null vector if *x* is None, or returns a null vector from base vector *x*, if *x*^self.I\_base ==0  
returns *x*,  
a null vector will lay on the horisphere

**clifford.cga.CGA.rotation**

**CGA.rotation** (\*args) → clifford.cga.Rotation  
see *Rotation*

**clifford.cga.CGA.round**

**CGA.round** (\*args) → clifford.cga.Round  
see *Round*

**clifford.cga.CGA.straight\_up**

`CGA.straight_up(x)` → `clifford._multivector.MultiVector`  
 place a vector from `layout_orig` into this CGA, without `up()`

**clifford.cga.CGA.translation**

`CGA.translation(*args)` → `clifford.cga.Translation`  
 see *Translation*

**clifford.cga.CGA.transversion**

`CGA.transversion(*args)` → `clifford.cga.Transversion`  
 see *Transversion*

**2.2.2 Objects**

<code>Flat(cga, *args)</code>	A line, plane, or hyperplane.
<code>Round(cga, *args)</code>	A point pair, circle, sphere or hyper-sphere.

**clifford.cga.Flat**

**class** `clifford.cga.Flat(cga, *args)`

A line, plane, or hyperplane.

Typically constructed as method of existing `cga`, like `cga.flat()`

multivector is accessible by `mv` property

**Parameters**

- **cga** (CGA) – the `cga` object
- **args** (`[int, Multivector, Multivectors]`) –
  - if nothing supplied, generate a flat of highest dimension
  - `int`: dimension of flat (2=line, 3=plane, etc)
  - `Multivector` : can be \* existing `Multivector` representing the Flat \* vectors on the flat

**Examples**

```
>>> cga = CGA(3)
>>> locals().update(cga.blades)
>>> F = cga.flat()           # from None
>>> F = cga.flat(2)          # from dim of space
>>> F = cga.flat(e1, e2)      # from points
>>> F = cga.flat(cga.flat().mv) # from existing multivector
```

## Methods

<code>__init__</code>	Initialize self.
<code>inverted</code>	inverted version of this thing.
<code>involuted</code>	inverted version of this thing.

### `clifford.cga.Flat.__init__`

`Flat.__init__(cga, *args) → None`  
Initialize self. See `help(type(self))` for accurate signature.

### `clifford.cga.Flat.inverted`

`Flat.inverted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
  
`self -> ep*self*ep`  
where `ep` is the positive added basis vector

### `clifford.cga.Flat.involuted`

`Flat.involuted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
  
`self -> E0*self*E0`  
where `E0` is the added minkowski bivector

### `clifford.cga.Round`

**class** `clifford.cga.Round(cga, *args)`  
A point pair, circle, sphere or hyper-sphere.  
Typically constructed as method of existing `cga`, like `cga.round()`  
multivector is accessible by `mv` property

#### Parameters

- **cga** (*CGA*) – the `cga` object
- **args** (`[int, Multivector, Multivectors]`) –
  - if nothing supplied, generate a round of highest dimension
  - `int`: dimension of flat (2=point pair, 3=circle, etc)
  - `Multivector` : can be \* existing `Multivector` representing the round \* vectors on the round

## Examples

```
>>> cga = CGA(3)
>>> locals().update(cga.blades)
>>> cga.round()           # from None
Sphere
>>> cga.round(2)         # from dim of space
Sphere
>>> cga.round(e1, e2, -e1) # from points
Circle
>>> cga.round(cga.flat().mv) # from existing multivector
Sphere
```

## Attributes

<code>center</code>	center of this round, as a null vector
<code>center_down</code>	center of this round, as a down-projected vector (in I_base)
<code>dim</code>	dimension of this round
<code>dual</code>	<code>self.mv* self.layout.I</code>
<code>radius</code>	radius of the round (a float)

### `clifford.cga.Round.center`

**property** `Round.center`  
center of this round, as a null vector

### `clifford.cga.Round.center_down`

**property** `Round.center_down`  
center of this round, as a down-projected vector (in I\_base)  
(but still in cga's layout)

### `clifford.cga.Round.dim`

**property** `Round.dim`  
dimension of this round

### `clifford.cga.Round.dual`

**property** `Round.dual`  
`self.mv* self.layout.I`

**clifford.cga.Round.radius**

**property** `Round.radius`  
radius of the round (a float)

**Methods**

<code>__init__</code>	Initialize self.
<code>from_center_radius</code>	construct a round from center/radius
<code>inverted</code>	inverted version of this thing.
<code>involuted</code>	inverted version of this thing.

**clifford.cga.Round.\_\_init\_\_**

`Round.__init__(cga, *args) → None`  
Initialize self. See `help(type(self))` for accurate signature.

**clifford.cga.Round.from\_center\_radius**

`Round.from_center_radius(center, radius)`  
construct a round from center/radius

**clifford.cga.Round.inverted**

`Round.inverted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
  
`self -> ep*self*ep`  
where `ep` is the positive added basis vector

**clifford.cga.Round.involuted**

`Round.involuted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
  
`self -> E0*self*E0`  
where `E0` is the added minkowski bivector

## 2.2.3 Operators

<i>Rotation</i> (cga, *args)	A Rotation
<i>Dilation</i> (cga, *args)	A global dilation
<i>Translation</i> (cga, *args)	A Translation
<i>Transversion</i> (cga, *args)	A Transversion

### clifford.cga.Rotation

**class** clifford.cga.**Rotation**(cga, \*args)

A Rotation

Can be constructed from a generator, rotor, or none

**Parameters** **args** ([none, *clifford.Multivector*]) – if none, a random translation will be generated  
several types of Multivectors can be used:

- bivector - interpreted as the generator
- existing translation rotor

### Examples

```
>>> cga = CGA(3)
>>> locals().update(cga.blades)
>>> R = cga.rotation()           # from None
>>> R = cga.rotation(e12+e23)    # from bivector
>>> R = cga.rotation(R.mv)      # from bivector
```

### Methods

<i>__init__</i>	Initialize self.
<i>inverted</i>	inverted version of this thing.
<i>involuted</i>	inverted version of this thing.

### clifford.cga.Rotation.\_\_init\_\_

Rotation.**\_\_init\_\_**(cga, \*args) → None

Initialize self. See help(type(self)) for accurate signature.

### clifford.cga.Rotation.inverted

`Rotation.inverted()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.

`self` → `ep*self*ep`

where `ep` is the positive added basis vector

### clifford.cga.Rotation.involuted

`Rotation.involuted()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.

`self` → `E0*self*E0`

where `E0` is the added minkowski bivector

### clifford.cga.Dilation

**class** `clifford.cga.Dilation(cga, *args)`

A global dilation

**Parameters** `args` (`[none, number]`) – if none, a random dilation will be generated if a number, dilation of given amount

### Examples

```
>>> cga = CGA(3)
>>> D = cga.dilation()           # from none
>>> D = cga.dilation(.4)        # from number
```

### Methods

<code>__init__</code>	Initialize self.
<code>inverted</code>	inverted version of this thing.
<code>involuted</code>	inverted version of this thing.

### clifford.cga.Dilation.\_\_init\_\_

`Dilation.__init__(cga, *args)` → `None`

Initialize self. See `help(type(self))` for accurate signature.



**clifford.cga.Dilation.inverted**

`Dilation.inverted()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.

`self` → `ep*self*ep`

where `ep` is the positive added basis vector

**clifford.cga.Dilation.involutd**

`Dilation.involutd()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.

`self` → `E0*self*E0`

where `E0` is the added minkowski bivector

**clifford.cga.Translation**

**class** `clifford.cga.Translation(cga, *args)`  
 A Translation

Can be constructed from a vector in base space or a null vector, or nothing.

**Parameters** `args` ([`none`, `clifford.Multivector`]) – if `none`, a random translation will be generated  
 several types of Multivectors can be used:

- base vector - vector in base space
- null vector
- existing translation rotor

**Examples**

```
>>> cga = CGA(3)
>>> locals().update(cga.blades)
>>> T = cga.translation()           # from None
>>> T = cga.translation(e1+e2)     # from base vector
>>> T = cga.translation(cga.up(e1+e2)) # from null vector
>>> T = cga.translation(T.mv)      # from existing translation rotor
```

**Methods**

<code>__init__</code>	Initialize self.
<code>inverted</code>	inverted version of this thing.
<code>involutd</code>	inverted version of this thing.

### clifford.cga.Translation.\_\_init\_\_

Translation.\_\_init\_\_(cga, \*args) → None  
Initialize self. See help(type(self)) for accurate signature.

### clifford.cga.Translation.inverted

Translation.inverted() → clifford.\_multivector.MultiVector  
inverted version of this thing.  
self ->  $ep * self * ep$   
where  $ep$  is the positive added basis vector

### clifford.cga.Translation.involuted

Translation.involuted() → clifford.\_multivector.MultiVector  
inverted version of this thing.  
self ->  $E0 * self * E0$   
where  $E0$  is the added minkowski bivector

### clifford.cga.Transversion

**class** clifford.cga.Transversion(cga, \*args)  
A Transversion

A transversion is a combination of an inversion-translation-inversion, or in other words an inverted translation operator. This inherits from *Translation*

Can be constructed from a vector in base space or a null vector, or nothing.

**Parameters** **args** ([none, clifford.Multivector]) – if none, a random transversion will be generated  
several types of Multivectors can be used:

- base vector - vector in base space
- null vector
- existing transversion rotor

### Examples

```
>>> cga = CGA(3)
>>> locals().update(cga.blades)
>>> K = cga.transversion()           # from None
>>> K = cga.transversion(e1+e2)      # from base vector
>>> K = cga.transversion(cga.up(e1+e2)) # from null vector
>>> T = cga.translation()
>>> K = cga.transversion(T.mv)      # from existing translation rotor
```

## Methods

<code>__init__</code>	Initialize self.
<code>inverted</code>	inverted version of this thing.
<code>involuted</code>	inverted version of this thing.

### `clifford.cga.Transversion.__init__`

`Transversion.__init__(cga, *args) → None`  
 Initialize self. See `help(type(self))` for accurate signature.

### `clifford.cga.Transversion.inverted`

`Transversion.inverted()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.  
 $\text{self} \rightarrow \text{ep} * \text{self} * \text{ep}$   
 where  $\text{ep}$  is the positive added basis vector

### `clifford.cga.Transversion.involuted`

`Transversion.involuted()` → `clifford._multivector.MultiVector`  
 inverted version of this thing.  
 $\text{self} \rightarrow \text{E0} * \text{self} * \text{E0}$   
 where  $\text{E0}$  is the added minkowski bivector

## 2.2.4 Meta-Class

<code>CGAThing(cga)</code>	base class for cga objects and operators.
----------------------------	---

### `clifford.cga.CGAThing`

**class** `clifford.cga.CGAThing(cga: clifford.cga.CGA)`  
 base class for cga objects and operators.  
 maps versor product to `__call__`.

## Methods

<code>__init__</code>	Initialize self.
<code>inverted</code>	inverted version of this thing.
<code>involuted</code>	inverted version of this thing.

### `clifford.cga.CGAThing.__init__`

`CGAThing.__init__(cga: clifford.cga.CGA) → None`  
Initialize self. See `help(type(self))` for accurate signature.

### `clifford.cga.CGAThing.inverted`

`CGAThing.inverted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
 $\text{self} \rightarrow \text{ep} * \text{self} * \text{ep}$   
where `ep` is the positive added basis vector

### `clifford.cga.CGAThing.involuted`

`CGAThing.involuted()` → `clifford._multivector.MultiVector`  
inverted version of this thing.  
 $\text{self} \rightarrow E0 * \text{self} * E0$   
where `E0` is the added minkowski bivector

## 2.3 tools (`clifford.tools`)

Algorithms and tools of various kinds.

### 2.3.1 Tools for specific ga's

<code>g3</code>	Tools for 3DGA ( <code>g3</code> )
<code>g3c</code>	Tools for 3DCGA ( <code>g3c</code> )

**clifford.tools.g3**

Tools for 3DGA (g3)

**3DGA Tools****Rotation Conversion Methods**

<i>quaternion_to_rotor</i> (quaternion)	Converts a quaternion into a pure rotation rotor
<i>rotor_to_quaternion</i> (R)	Converts a pure rotation rotor into a quaternion
<i>quaternion_to_matrix</i> (q)	Converts a quaternion into a rotation matrix
<i>rotation_matrix_to_quaternion</i> (a)	Converts a rotation matrix into a quaternion

**clifford.tools.g3.quaternion\_to\_rotor**

`clifford.tools.g3.quaternion_to_rotor` (*quaternion*)  
Converts a quaternion into a pure rotation rotor

**clifford.tools.g3.rotor\_to\_quaternion**

`clifford.tools.g3.rotor_to_quaternion` (*R*)  
Converts a pure rotation rotor into a quaternion

**clifford.tools.g3.quaternion\_to\_matrix**

`clifford.tools.g3.quaternion_to_matrix` (*q*)  
Converts a quaternion into a rotation matrix

**clifford.tools.g3.rotation\_matrix\_to\_quaternion**

`clifford.tools.g3.rotation_matrix_to_quaternion` (*a*)  
Converts a rotation matrix into a quaternion

**Generation Methods**

<i>random_unit_vector</i> ()	Creates a random unit vector
<i>random_euc_mv</i> ([l_max])	Creates a random vector normally distributed with length l_max
<i>generate_rotation_rotor</i> (theta, euc_vector_m, ...)	Generates a rotation of angle theta in the m, n plane
<i>random_rotation_rotor</i> ([max_angle])	Creates a random rotation rotor

**clifford.tools.g3.random\_unit\_vector**

`clifford.tools.g3.random_unit_vector()`  
Creates a random unit vector

**clifford.tools.g3.random\_euc\_mv**

`clifford.tools.g3.random_euc_mv(l_max=10)`  
Creates a random vector normally distributed with length l\_max

**clifford.tools.g3.generate\_rotation\_rotor**

`clifford.tools.g3.generate_rotation_rotor(theta, euc_vector_m, euc_vector_n)`  
Generates a rotation of angle theta in the m, n plane

**clifford.tools.g3.random\_rotation\_rotor**

`clifford.tools.g3.random_rotation_rotor(max_angle=3.141592653589793)`  
Creates a random rotation rotor

**Misc**

<code>angle_between_vectors(v1, v2)</code>	Returns the angle between two conformal vectors
<code>np_to_euc_mv(np_in)</code>	Converts a 3d numpy vector to a 3d GA point
<code>euc_mv_to_np(euc_point)</code>	Converts a 3d GA point to a 3d numpy vector
<code>euc_cross_prod(euc_a, euc_b)</code>	Implements the cross product in GA
<code>rotor_vector_to_vector(v1, v2)</code>	Creates a rotor that takes one vector into another
<code>correlation_matrix(u_list, v_list)</code>	Creates a correlation matrix between vector lists
<code>GA_SVD(u_list, v_list)</code>	Does SVD on a pair of GA vectors
<code>rotation_matrix_align_vecs(u_list, v_list)</code>	Returns the rotation matrix that aligns the set of vectors u and v
<code>rotor_align_vecs(u_list, v_list)</code>	Returns the rotation rotor that aligns the set of vectors u and v

**clifford.tools.g3.angle\_between\_vectors**

`clifford.tools.g3.angle_between_vectors(v1, v2)`  
Returns the angle between two conformal vectors

**clifford.tools.g3.np\_to\_euc\_mv**

`clifford.tools.g3.np_to_euc_mv(np_in)`  
Converts a 3d numpy vector to a 3d GA point

**clifford.tools.g3.euc\_mv\_to\_np**

`clifford.tools.g3.euc_mv_to_np(euc_point)`  
Converts a 3d GA point to a 3d numpy vector

**clifford.tools.g3.euc\_cross\_prod**

`clifford.tools.g3.euc_cross_prod(euc_a, euc_b)`  
Implements the cross product in GA

**clifford.tools.g3.rotor\_vector\_to\_vector**

`clifford.tools.g3.rotor_vector_to_vector(v1, v2)`  
Creates a rotor that takes one vector into another

**clifford.tools.g3.correlation\_matrix**

`clifford.tools.g3.correlation_matrix(u_list, v_list)`  
Creates a correlation matrix between vector lists

**clifford.tools.g3.GA\_SVD**

`clifford.tools.g3.GA_SVD(u_list, v_list)`  
Does SVD on a pair of GA vectors

**clifford.tools.g3.rotation\_matrix\_align\_vecs**

`clifford.tools.g3.rotation_matrix_align_vecs(u_list, v_list)`  
Returns the rotation matrix that aligns the set of vectors u and v

**clifford.tools.g3.rotor\_align\_vecs**

`clifford.tools.g3.rotor_align_vecs(u_list, v_list)`  
Returns the rotation rotor that aligns the set of vectors u and v

## clifford.tools.g3c

Tools for 3DCGA (g3c)

### 3DCGA Tools

#### Generation Methods

<code>random_bivector()</code>	Creates a random bivector on the form described by R.
<code>standard_point_pair_at_origin()</code>	Creates a standard point pair at the origin
<code>random_point_pair_at_origin()</code>	Creates a random point pair bivector object at the origin
<code>random_point_pair()</code>	Creates a random point pair bivector object
<code>standard_line_at_origin()</code>	Creates a standard line at the origin
<code>random_line_at_origin()</code>	Creates a random line at the origin
<code>random_line()</code>	Creates a random line
<code>random_circle_at_origin()</code>	Creates a random circle at the origin
<code>random_circle()</code>	Creates a random circle
<code>random_sphere_at_origin()</code>	Creates a random sphere at the origin
<code>random_sphere()</code>	Creates a random sphere
<code>random_plane_at_origin()</code>	Creates a random plane at the origin
<code>random_plane()</code>	Creates a random plane
<code>generate_n_clusters(object_generator, ...)</code>	Creates n_clusters of random objects
<code>generate_random_object_cluster(n_objects, ...)</code>	Creates a cluster of random objects
<code>random_translation_rotor([maximum_translation])</code>	generate a random translation rotor
<code>random_rotation_translation_rotor([...])</code>	generate a random combined rotation and translation rotor
<code>random_conformal_point([l_max])</code>	Creates a random conformal point
<code>generate_dilation_rotor(scale)</code>	Generates a rotor that performs dilation about the origin
<code>generate_translation_rotor(euc_vector_a)</code>	Generates a rotor that translates objects along the euclidean vector euc_vector_a

## clifford.tools.g3c.random\_bivector

`clifford.tools.g3c.random_bivector()`

Creates a random bivector on the form described by R. Wareham in Mesh Vertex Pose and Position Interpolation using Geometric Algebra.  $B = ab + c \cdot n_{\infty}$  where  $a, b, c$  in  $\text{mathcal{R}}^3$



**clifford.tools.g3c.standard\_point\_pair\_at\_origin**

`clifford.tools.g3c.standard_point_pair_at_origin()`  
Creates a standard point pair at the origin

**clifford.tools.g3c.random\_point\_pair\_at\_origin**

`clifford.tools.g3c.random_point_pair_at_origin()`  
Creates a random point pair bivector object at the origin

**clifford.tools.g3c.random\_point\_pair**

`clifford.tools.g3c.random_point_pair()`  
Creates a random point pair bivector object

**clifford.tools.g3c.standard\_line\_at\_origin**

`clifford.tools.g3c.standard_line_at_origin()`  
Creates a standard line at the origin

**clifford.tools.g3c.random\_line\_at\_origin**

`clifford.tools.g3c.random_line_at_origin()`  
Creates a random line at the origin

**clifford.tools.g3c.random\_line**

`clifford.tools.g3c.random_line()`  
Creates a random line

**clifford.tools.g3c.random\_circle\_at\_origin**

`clifford.tools.g3c.random_circle_at_origin()`  
Creates a random circle at the origin

**clifford.tools.g3c.random\_circle**

`clifford.tools.g3c.random_circle()`  
Creates a random circle

### **clifford.tools.g3c.random\_sphere\_at\_origin**

`clifford.tools.g3c.random_sphere_at_origin()`  
Creates a random sphere at the origin

### **clifford.tools.g3c.random\_sphere**

`clifford.tools.g3c.random_sphere()`  
Creates a random sphere

### **clifford.tools.g3c.random\_plane\_at\_origin**

`clifford.tools.g3c.random_plane_at_origin()`  
Creates a random plane at the origin

### **clifford.tools.g3c.random\_plane**

`clifford.tools.g3c.random_plane()`  
Creates a random plane

### **clifford.tools.g3c.generate\_n\_clusters**

`clifford.tools.g3c.generate_n_clusters(object_generator, n_clusters, n_objects_per_cluster)`  
Creates `n_clusters` of random objects

### **clifford.tools.g3c.generate\_random\_object\_cluster**

`clifford.tools.g3c.generate_random_object_cluster(n_objects, object_generator, max_cluster_trans=1.0, max_cluster_rot=0.39269908169872414)`  
Creates a cluster of random objects

### **clifford.tools.g3c.random\_translation\_rotor**

`clifford.tools.g3c.random_translation_rotor(maximum_translation=10.0)`  
generate a random translation rotor

**clifford.tools.g3c.random\_rotation\_translation\_rotor**

`clifford.tools.g3c.random_rotation_translation_rotor` (*maximum\_translation=10.0*,  
*maximum\_angle=3.141592653589793*)  
 generate a random combined rotation and translation rotor

**clifford.tools.g3c.random\_conformal\_point**

`clifford.tools.g3c.random_conformal_point` (*l\_max=10*)  
 Creates a random conformal point

**clifford.tools.g3c.generate\_dilation\_rotor**

`clifford.tools.g3c.generate_dilation_rotor` (*scale*)  
 Generates a rotor that performs dilation about the origin

**clifford.tools.g3c.generate\_translation\_rotor**

`clifford.tools.g3c.generate_translation_rotor` (*euc\_vector\_a*)  
 Generates a rotor that translates objects along the euclidean vector *euc\_vector\_a*

**Geometry Methods**

<code>intersect_line_and_plane_to_point</code> (line, plane)	Returns the point at the intersection of a line and plane If there is no intersection it returns None
<code>val_intersect_line_and_plane_to_point</code> (...)	
<code>quaternion_and_vector_to_rotor</code> (quaternion, ...)	Takes in a quaternion and a vector and returns a conformal rotor that implements the transformation
<code>get_center_from_sphere</code> (sphere)	Returns the conformal point at the centre of a sphere by reflecting the point at infinity
<code>get_radius_from_sphere</code> (sphere)	Returns the radius of a sphere
<code>point_pair_to_end_points</code> (T)	Extracts the end points of a point pair bivector
<code>val_point_pair_to_end_points</code> (T)	
<code>get_circle_in_euc</code> (circle)	Extracts all the normal stuff for a circle
<code>circle_to_sphere</code> (C)	returns the sphere for which the input circle is the perimeter
<code>line_to_point_and_direction</code> (line)	converts a line to the conformal nearest point to the origin and a euc direction vector in direction of the line
<code>get_plane_origin_distance</code> (plane)	Get the distance between a given plane and the origin
<code>get_plane_normal</code> (plane)	Get the normal to the plane
<code>get_nearest_plane_point</code> (plane)	Get the nearest point to the origin on the plane
<code>val_convert_2D_polar_line_to_conformal_line</code> (C)	Converts a 2D polar line to a conformal line
<code>convert_2D_polar_line_to_conformal_line</code> (C, ...)	Converts a 2D polar line to a conformal line

continues on next page

Table 25 – continued from previous page

<code>val_convert_2D_point_to_conformal(x, y)</code>	
<code>convert_2D_point_to_conformal(x, y)</code>	Convert a 2D point to conformal
<code>val_distance_point_to_line(point, line)</code>	Returns the euclidean distance between a point and a line
<code>distance_polar_line_to_euc_point_2d(rho, ...)</code>	Return the distance between a polar line and a euclidean point in 2D
<code>midpoint_between_lines(L1, L2)</code>	Gets the point that is maximally close to both lines Hadfield and Lasenby AGACSE2018
<code>val_midpoint_between_lines(L1_val, L2_val)</code>	
<code>midpoint_of_line_cluster(line_cluster)</code>	Gets a center point of a line cluster Hadfield and Lasenby AGACSE2018
<code>val_midpoint_of_line_cluster(array_line_cluster)</code>	Gets a center point of a line cluster Hadfield and Lasenby AGACSE2018
<code>val_midpoint_of_line_cluster_grad(...)</code>	Gets an approximate center point of a line cluster Hadfield and Lasenby AGACSE2018
<code>get_line_intersection(L3, Ldd)</code>	Gets the point of intersection of two orthogonal lines that meet $X_{dd} = L_{dd} * no * L_{dd} + no$ $X_{ddd} = L_3 * X_{dd} * L_3$ $P_d = 0.5 * (X_{dd} + X_{ddd})$ $P = - (P_d * ninf * P_d) / (1 / (2 * (P_d linf)^2)) [0]$
<code>val_get_line_intersection(L3_val, Ldd_val)</code>	
<code>project_points_to_plane(point_list, plane)</code>	Takes a load of points and projects them onto a plane
<code>project_points_to_sphere(point_list, sphere)</code>	Takes a load of points and projects them onto a sphere
<code>project_points_to_circle(point_list, circle)</code>	Takes a load of point and projects them onto a circle The closest flag determines if it should be the closest or furthest point on the circle
<code>project_points_to_line(point_list, line)</code>	Takes a load of points and projects them onto a line
<code>iterative_closest_points_on_circles(C1, C2)</code>	Given two circles C1 and C2 this calculates the closest points on each of them to the other
<code>closest_point_on_line_from_circle(C, L[, eps])</code>	Returns the point on the line L that is closest to the circle C Uses the algorithm described in Appendix A of Andreas Aristidou's PhD thesis
<code>closest_point_on_circle_from_line(C, L[, eps])</code>	Returns the point on the circle C that is closest to the line L Uses the algorithm described in Appendix A of Andreas Aristidou's PhD thesis
<code>iterative_closest_points_circle_line(C, L[, ...])</code>	Given a circle C and line L this calculates the closest points on each of them to the other.
<code>iterative_furthest_points_on_circles(C1, C2)</code>	Given two circles C1 and C2 this calculates the closest points on each of them to the other
<code>sphere_beyond_plane(sphere, plane)</code>	Check if the sphere is fully beyond the plane in the direction of the plane normal
<code>sphere_behind_plane(sphere, plane)</code>	Check if the sphere is fully behind the plane in the direction of the plane normal, ie the opposite of sphere_beyond_plane

**clifford.tools.g3c.intersect\_line\_and\_plane\_to\_point**

`clifford.tools.g3c.intersect_line_and_plane_to_point` (*line*, *plane*)

Returns the point at the intersection of a line and plane. If there is no intersection it returns None.

**clifford.tools.g3c.val\_intersect\_line\_and\_plane\_to\_point**

`clifford.tools.g3c.val_intersect_line_and_plane_to_point` (*line\_val*, *plane\_val*)

**clifford.tools.g3c.quaternion\_and\_vector\_to\_rotor**

`clifford.tools.g3c.quaternion_and_vector_to_rotor` (*quaternion*, *vector*)

Takes in a quaternion and a vector and returns a conformal rotor that implements the transformation.

**clifford.tools.g3c.get\_center\_from\_sphere**

`clifford.tools.g3c.get_center_from_sphere` (*sphere*)

Returns the conformal point at the centre of a sphere by reflecting the point at infinity.

**clifford.tools.g3c.get\_radius\_from\_sphere**

`clifford.tools.g3c.get_radius_from_sphere` (*sphere*)

Returns the radius of a sphere.

**clifford.tools.g3c.point\_pair\_to\_end\_points**

`clifford.tools.g3c.point_pair_to_end_points` (*T*)

Extracts the end points of a point pair bivector.

**clifford.tools.g3c.val\_point\_pair\_to\_end\_points**

`clifford.tools.g3c.val_point_pair_to_end_points` (*T*)

**clifford.tools.g3c.get\_circle\_in\_euc**

`clifford.tools.g3c.get_circle_in_euc` (*circle*)

Extracts all the normal stuff for a circle.

### **clifford.tools.g3c.circle\_to\_sphere**

`clifford.tools.g3c.circle_to_sphere(C)`  
returns the sphere for which the input circle is the perimeter

### **clifford.tools.g3c.line\_to\_point\_and\_direction**

`clifford.tools.g3c.line_to_point_and_direction(line)`  
converts a line to the conformal nearest point to the origin and a euc direction vector in direction of the line

### **clifford.tools.g3c.get\_plane\_origin\_distance**

`clifford.tools.g3c.get_plane_origin_distance(plane)`  
Get the distance between a given plane and the origin

### **clifford.tools.g3c.get\_plane\_normal**

`clifford.tools.g3c.get_plane_normal(plane)`  
Get the normal to the plane

### **clifford.tools.g3c.get\_nearest\_plane\_point**

`clifford.tools.g3c.get_nearest_plane_point(plane)`  
Get the nearest point to the origin on the plane

### **clifford.tools.g3c.val\_convert\_2D\_polar\_line\_to\_conformal\_line**

`clifford.tools.g3c.val_convert_2D_polar_line_to_conformal_line(rho, theta)`  
Converts a 2D polar line to a conformal line

### **clifford.tools.g3c.convert\_2D\_polar\_line\_to\_conformal\_line**

`clifford.tools.g3c.convert_2D_polar_line_to_conformal_line(rho, theta)`  
Converts a 2D polar line to a conformal line

### **clifford.tools.g3c.val\_convert\_2D\_point\_to\_conformal**

`clifford.tools.g3c.val_convert_2D_point_to_conformal(x, y)`

**clifford.tools.g3c.convert\_2D\_point\_to\_conformal**

`clifford.tools.g3c.convert_2D_point_to_conformal(x, y)`  
Convert a 2D point to conformal

**clifford.tools.g3c.val\_distance\_point\_to\_line**

`clifford.tools.g3c.val_distance_point_to_line(point, line)`  
Returns the euclidean distance between a point and a line

**clifford.tools.g3c.distance\_polar\_line\_to\_euc\_point\_2d**

`clifford.tools.g3c.distance_polar_line_to_euc_point_2d(rho, theta, x, y)`  
Return the distance between a polar line and a euclidean point in 2D

**clifford.tools.g3c.midpoint\_between\_lines**

`clifford.tools.g3c.midpoint_between_lines(L1, L2)`  
Gets the point that is maximally close to both lines Hadfield and Lasenby AGACSE2018

**clifford.tools.g3c.val\_midpoint\_between\_lines**

`clifford.tools.g3c.val_midpoint_between_lines(L1_val, L2_val)`

**clifford.tools.g3c.midpoint\_of\_line\_cluster**

`clifford.tools.g3c.midpoint_of_line_cluster(line_cluster)`  
Gets a center point of a line cluster Hadfield and Lasenby AGACSE2018

**clifford.tools.g3c.val\_midpoint\_of\_line\_cluster**

`clifford.tools.g3c.val_midpoint_of_line_cluster(array_line_cluster)`  
Gets a center point of a line cluster Hadfield and Lasenby AGACSE2018

**clifford.tools.g3c.val\_midpoint\_of\_line\_cluster\_grad**

`clifford.tools.g3c.val_midpoint_of_line_cluster_grad(array_line_cluster)`  
Gets an approximate center point of a line cluster Hadfield and Lasenby AGACSE2018

### **clifford.tools.g3c.get\_line\_intersection**

`clifford.tools.g3c.get_line_intersection(L3, Ldd)`

Gets the point of intersection of two orthogonal lines that meet  $X_{dd} = L_{dd} * no * L_{dd} + no$   $X_{ddd} = L_3 * X_{dd} * L_3$   
 $P_d = 0.5 * (X_{dd} + X_{ddd})$   $P = -(P_d * ninf * P_d)(1)/(2 * (P_dleinf)**2)[0]$

### **clifford.tools.g3c.val\_get\_line\_intersection**

`clifford.tools.g3c.val_get_line_intersection(L3_val, Ldd_val)`

### **clifford.tools.g3c.project\_points\_to\_plane**

`clifford.tools.g3c.project_points_to_plane(point_list, plane)`

Takes a load of points and projects them onto a plane

### **clifford.tools.g3c.project\_points\_to\_sphere**

`clifford.tools.g3c.project_points_to_sphere(point_list, sphere, closest=True)`

Takes a load of points and projects them onto a sphere

### **clifford.tools.g3c.project\_points\_to\_circle**

`clifford.tools.g3c.project_points_to_circle(point_list, circle, closest=True)`

Takes a load of point and projects them onto a circle The closest flag determines if it should be the closest or furthest point on the circle

### **clifford.tools.g3c.project\_points\_to\_line**

`clifford.tools.g3c.project_points_to_line(point_list, line)`

Takes a load of points and projects them onto a line

### **clifford.tools.g3c.iterative\_closest\_points\_on\_circles**

`clifford.tools.g3c.iterative_closest_points_on_circles(C1, C2, niterations=20)`

Given two circles C1 and C2 this calculates the closest points on each of them to the other

Changed in version 1.3: Renamed from `closest_points_on_circles`



**clifford.tools.g3c.closest\_point\_on\_line\_from\_circle**

`clifford.tools.g3c.closest_point_on_line_from_circle(C, L, eps=1e-06)`

Returns the point on the line L that is closest to the circle C Uses the algorithm described in Appendix A of Andreas Aristidou's PhD thesis

New in version 1.3.

**clifford.tools.g3c.closest\_point\_on\_circle\_from\_line**

`clifford.tools.g3c.closest_point_on_circle_from_line(C, L, eps=1e-06)`

Returns the point on the circle C that is closest to the line L Uses the algorithm described in Appendix A of Andreas Aristidou's PhD thesis

New in version 1.3.

**clifford.tools.g3c.iterative\_closest\_points\_circle\_line**

`clifford.tools.g3c.iterative_closest_points_circle_line(C, L, niterations=20)`

Given a circle C and line L this calculates the closest points on each of them to the other.

This is an iterative algorithm based on heuristics Nonetheless it appears to give results on par with `closest_point_on_circle_from_line()`.

Changed in version 1.3: Renamed from `closest_points_circle_line`

**clifford.tools.g3c.iterative\_furthest\_points\_on\_circles**

`clifford.tools.g3c.iterative_furthest_points_on_circles(C1, C2, niterations=20)`

Given two circles C1 and C2 this calculates the closest points on each of them to the other

Changed in version 1.3: Renamed from `furthest_points_on_circles`

**clifford.tools.g3c.sphere\_beyond\_plane**

`clifford.tools.g3c.sphere_beyond_plane(sphere, plane)`

Check if the sphere is fully beyond the plane in the direction of the plane normal

**clifford.tools.g3c.sphere\_behind\_plane**

`clifford.tools.g3c.sphere_behind_plane(sphere, plane)`

Check if the sphere is fully behind the plane in the direction of the plane normal, ie the opposite of `sphere_beyond_plane`

## Misc

<code>meet_val(a_val, b_val)</code>	
<code>meet(A, B)</code>	The meet algorithm as described in [LLW04].
<code>normalise_n_minus_1(mv)</code>	Normalises a conformal point so that it has an inner product of -1 with einf
<code>val_normalise_n_minus_1(mv_val)</code>	
<code>val_apply_rotor(mv_val, rotor_val)</code>	
<code>apply_rotor(mv_in, rotor)</code>	Applies rotor to multivector in a fast way
<code>val_apply_rotor_inv(mv_val, rotor_val, ...)</code>	
<code>apply_rotor_inv(mv_in, rotor, rotor_inv)</code>	Applies rotor to multivector in a fast way takes pre computed adjoint
<code>euc_dist(conf_mv_a, conf_mv_b)</code>	Returns the distance between two conformal points
<code>mult_with_ninf(mv)</code>	Convenience function for multiplication with ninf
<code>val_norm(mv_val)</code>	
<code>norm(mv)</code>	Alias of <code>clifford.MultiVector.__abs__()</code>
<code>val_normalised(mv_val)</code>	
<code>normalised(mv)</code>	Alias of <code>clifford.MultiVector.normal()</code>
<code>val_up(mv_val)</code>	
<code>fast_up(mv)</code>	Fast jitted up mapping
<code>val_normalInv(mv_val)</code>	
<code>val_homo(mv_val)</code>	
<code>val_down(mv_val)</code>	
<code>fast_down(mv)</code>	A fast version of down()
<code>dual_func(a_val)</code>	
<code>fast_dual(a)</code>	Fast dual
<code>disturb_object(mv_object[, ...])</code>	Disturbs an object by a random rotor
<code>project_val(val, grade)</code>	
<code>get_line_reflection_matrix(lines[, n_power])</code>	Generates the matrix that sums the reflection of a point in many lines
<code>val_get_line_reflection_matrix(line_array, ...)</code>	Generates the matrix that sums the reflection of a point in many lines
<code>val_truncated_get_line_reflection_matrix(line_array, ...)</code>	Generates the truncated matrix that sums the reflection of a point in many lines
<code>interpret_multivector_as_object(mv)</code>	Takes an input multivector and returns what kind of object it is

continues on next page

Table 26 – continued from previous page

<code>normalise_TR_to_unit_T(TR)</code>	Takes in a TR rotor extracts the R and T normalises the T to unit displacement magnitude rebuilds the TR rotor with the new displacement rotor returns the new TR and the original length of the T rotor
<code>scale_TR_translation(TR, scale)</code>	Takes in a TR rotor and a scale extracts the R and T scales the T displacement magnitude by scale rebuilds the TR rotor with the new displacement rotor returns the new TR rotor
<code>val_unsign_sphere(S)</code>	

**clifford.tools.g3c.meet\_val**

`clifford.tools.g3c.meet_val(a_val, b_val)`

**clifford.tools.g3c.meet**

`clifford.tools.g3c.meet(A, B)`

The meet algorithm as described in [LLW04].

$$I5 * ((I5 * A) \wedge (I5 * B))$$
**clifford.tools.g3c.normalise\_n\_minus\_1**

`clifford.tools.g3c.normalise_n_minus_1(mv)`

Normalises a conformal point so that it has an inner product of -1 with einf

**clifford.tools.g3c.val\_normalise\_n\_minus\_1**

`clifford.tools.g3c.val_normalise_n_minus_1(mv_val)`

**clifford.tools.g3c.val\_apply\_rotor**

`clifford.tools.g3c.val_apply_rotor(mv_val, rotor_val)`

**clifford.tools.g3c.apply\_rotor**

`clifford.tools.g3c.apply_rotor(mv_in, rotor)`

Applies rotor to multivector in a fast way

### **clifford.tools.g3c.val\_apply\_rotor\_inv**

`clifford.tools.g3c.val_apply_rotor_inv(mv_val, rotor_val, rotor_val_inv)`

### **clifford.tools.g3c.apply\_rotor\_inv**

`clifford.tools.g3c.apply_rotor_inv(mv_in, rotor, rotor_inv)`  
Applies rotor to multivector in a fast way takes pre computed adjoint

### **clifford.tools.g3c.euc\_dist**

`clifford.tools.g3c.euc_dist(conf_mv_a, conf_mv_b)`  
Returns the distance between two conformal points

### **clifford.tools.g3c.mult\_with\_ninf**

`clifford.tools.g3c.mult_with_ninf(mv)`  
Convenience function for multiplication with ninf

### **clifford.tools.g3c.val\_norm**

`clifford.tools.g3c.val_norm(mv_val)`

### **clifford.tools.g3c.norm**

`clifford.tools.g3c.norm(mv)`  
Alias of `clifford.MultiVector.__abs__()`

### **clifford.tools.g3c.val\_normalised**

`clifford.tools.g3c.val_normalised(mv_val)`

### **clifford.tools.g3c.normalised**

`clifford.tools.g3c.normalised(mv)`  
Alias of `clifford.MultiVector.normal()`

**clifford.tools.g3c.val\_up**

`clifford.tools.g3c.val_up(mv_val)`

**clifford.tools.g3c.fast\_up**

`clifford.tools.g3c.fast_up(mv)`  
Fast jitted up mapping

**clifford.tools.g3c.val\_normalInv**

`clifford.tools.g3c.val_normalInv(mv_val)`

**clifford.tools.g3c.val\_homo**

`clifford.tools.g3c.val_homo(mv_val)`

**clifford.tools.g3c.val\_down**

`clifford.tools.g3c.val_down(mv_val)`

**clifford.tools.g3c.fast\_down**

`clifford.tools.g3c.fast_down(mv)`  
A fast version of down()

**clifford.tools.g3c.dual\_func**

`clifford.tools.g3c.dual_func(a_val)`

**clifford.tools.g3c.fast\_dual**

`clifford.tools.g3c.fast_dual(a)`  
Fast dual

**clifford.tools.g3c.disturb\_object**

`clifford.tools.g3c.disturb_object(mv_object, maximum_translation=0.01, maxi-  
mum_angle=0.01)`  
Disturbs an object by a random rotor

### **clifford.tools.g3c.project\_val**

`clifford.tools.g3c.project_val(val, grade)`

### **clifford.tools.g3c.get\_line\_reflection\_matrix**

`clifford.tools.g3c.get_line_reflection_matrix(lines, n_power=1)`  
 Generates the matrix that sums the reflection of a point in many lines

### **clifford.tools.g3c.val\_get\_line\_reflection\_matrix**

`clifford.tools.g3c.val_get_line_reflection_matrix(line_array: numpy.ndarray,  
 n_power: int) → numpy.ndarray`  
 Generates the matrix that sums the reflection of a point in many lines

### **clifford.tools.g3c.val\_truncated\_get\_line\_reflection\_matrix**

`clifford.tools.g3c.val_truncated_get_line_reflection_matrix(line_array:  
numpy.ndarray,  
 n_power: int) → numpy.ndarray`  
 Generates the truncated matrix that sums the reflection of a point in many lines

### **clifford.tools.g3c.interpret\_multivector\_as\_object**

`clifford.tools.g3c.interpret_multivector_as_object(mv)`  
 Takes an input multivector and returns what kind of object it is  
 -1 -> not a blade 0 -> a 1 vector but not a point 1 -> a euclidean point 2 -> a conformal point 3 -> a point pair 4  
 -> a circle 5 -> a line 6 -> a sphere 7 -> a plane  
 Similar to `clifford.tools.classify.classify()`, although that function does a little more work in  
 order to produce full characterizations.

### **clifford.tools.g3c.normalise\_TR\_to\_unit\_T**

`clifford.tools.g3c.normalise_TR_to_unit_T(TR)`  
 Takes in a TR rotor extracts the R and T normalises the T to unit displacement magnitude rebuilds the TR rotor  
 with the new displacement rotor returns the new TR and the original length of the T rotor

**clifford.tools.g3c.scale\_TR\_translation**

`clifford.tools.g3c.scale_TR_translation` (*TR, scale*)

Takes in a TR rotor and a scale extracts the R and T scales the T displacement magnitude by scale rebuilds the TR rotor with the new displacement rotor returns the new TR rotor

**clifford.tools.g3c.val\_unsign\_sphere**

`clifford.tools.g3c.val_unsign_sphere` (*S*)

**Root Finding**

<code>dorst_norm_val(sigma_val)</code>	
<code>check_sigma_for_positive_root_val(sigma_val)</code>	
<code>check_sigma_for_positive_root(sigma)</code>	Square Root of Rotors - Checks for a positive root
<code>check_sigma_for_negative_root_val(sigma_value)</code>	
<code>check_sigma_for_negative_root(sigma)</code>	Square Root of Rotors - Checks for a negative root
<code>check_infinite_roots_val(sigma_value)</code>	
<code>check_infinite_roots(sigma)</code>	Square Root of Rotors - Checks for a infinite roots
<code>positive_root_val(sigma_val)</code>	
<code>negative_root_val(sigma_val)</code>	
<code>positive_root(sigma)</code>	Square Root of Rotors - Evaluates the positive root
<code>negative_root(sigma)</code>	Square Root of Rotors - Evaluates the negative root
<code>general_root_val(sigma_value)</code>	
<code>general_root(sigma)</code>	The general case of the root of a grade 0, 4 multivector
<code>val_annihilate_k(K_val, C_val)</code>	
<code>annihilate_k(K, C)</code>	Removes K from C = KX via (K[0] - K[4])*C
<code>pos_twiddle_root_val(C_value)</code>	
<code>neg_twiddle_root_val(C_value)</code>	
<code>pos_twiddle_root(C)</code>	Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg
<code>neg_twiddle_root(C)</code>	Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg
<code>square_roots_of_rotor(R)</code>	Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg
<code>n_th_rotor_root(R, n)</code>	Takes the n_th root of rotor R n must be a power of 2

continues on next page

Table 27 – continued from previous page

<i>interp_objects_root</i> (C1, C2, alpha)	Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly linearly interpolates conformal objects Return a valid object from the addition result C
<i>general_object_interpolation</i> (...[, kind])	Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 This is a general interpolation through the
<i>average_objects</i> (obj_list[, weights, ...])	Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C
<i>val_average_objects_with_weights</i> (obj_array, ...)	Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C
<i>val_average_objects</i> (obj_array)	Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C
<i>rotor_between_objects</i> (X1, X2)	Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a rotor that takes X1 to X2 Return a valid object from the addition result $1 + \gamma X_2 X_1$
<i>val_rotor_between_objects_root</i> (X1, X2)	
<i>val_rotor_between_objects_explicit</i> (X1, X2)	Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a rotor that takes X1 to X2
<i>calculate_S_over_mu</i> (X1, X2)	Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a factor that corrects the $X_1 + X_2$ back to a blade
<i>val_rotor_between_lines</i> (L1_val, L2_val)	Implements a very optimised rotor line to line extraction
<i>rotor_between_lines</i> (L1, L2)	Implements a very optimised rotor line to line extraction
<i>rotor_between_planes</i> (P1, P2)	return the rotor between two planes
<i>val_rotor_rotor_between_planes</i> (P1_val, P2_val)	

**clifford.tools.g3c.dorst\_norm\_val**

`clifford.tools.g3c.dorst_norm_val` (*sigma\_val*)



**clifford.tools.g3c.check\_sigma\_for\_positive\_root\_val**

`clifford.tools.g3c.check_sigma_for_positive_root_val` (*sigma\_val*)

**clifford.tools.g3c.check\_sigma\_for\_positive\_root**

`clifford.tools.g3c.check_sigma_for_positive_root` (*sigma*)

Square Root of Rotors - Checks for a positive root

**clifford.tools.g3c.check\_sigma\_for\_negative\_root\_val**

`clifford.tools.g3c.check_sigma_for_negative_root_val` (*sigma\_value*)

**clifford.tools.g3c.check\_sigma\_for\_negative\_root**

`clifford.tools.g3c.check_sigma_for_negative_root` (*sigma*)

Square Root of Rotors - Checks for a negative root

**clifford.tools.g3c.check\_infinite\_roots\_val**

`clifford.tools.g3c.check_infinite_roots_val` (*sigma\_value*)

**clifford.tools.g3c.check\_infinite\_roots**

`clifford.tools.g3c.check_infinite_roots` (*sigma*)

Square Root of Rotors - Checks for a infinite roots

**clifford.tools.g3c.positive\_root\_val**

`clifford.tools.g3c.positive_root_val` (*sigma\_val*)

**clifford.tools.g3c.negative\_root\_val**

`clifford.tools.g3c.negative_root_val` (*sigma\_val*)

**clifford.tools.g3c.positive\_root**

`clifford.tools.g3c.positive_root` (*sigma*)

Square Root of Rotors - Evaluates the positive root

### **clifford.tools.g3c.negative\_root**

`clifford.tools.g3c.negative_root` (*sigma*)  
Square Root of Rotors - Evaluates the negative root

### **clifford.tools.g3c.general\_root\_val**

`clifford.tools.g3c.general_root_val` (*sigma\_value*)

### **clifford.tools.g3c.general\_root**

`clifford.tools.g3c.general_root` (*sigma*)  
The general case of the root of a grade 0, 4 multivector  
Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg

### **clifford.tools.g3c.val\_annihilate\_k**

`clifford.tools.g3c.val_annihilate_k` (*K\_val*, *C\_val*)

### **clifford.tools.g3c.annihilate\_k**

`clifford.tools.g3c.annihilate_k` (*K*, *C*)  
Removes K from C = KX via (K[0] - K[4])\*C

### **clifford.tools.g3c.pos\_twiddle\_root\_val**

`clifford.tools.g3c.pos_twiddle_root_val` (*C\_value*)

### **clifford.tools.g3c.neg\_twiddle\_root\_val**

`clifford.tools.g3c.neg_twiddle_root_val` (*C\_value*)

### **clifford.tools.g3c.pos\_twiddle\_root**

`clifford.tools.g3c.pos_twiddle_root` (*C*)  
Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg

**clifford.tools.g3c.neg\_twiddle\_root**

`clifford.tools.g3c.neg_twiddle_root(C)`

Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg

**clifford.tools.g3c.square\_roots\_of\_rotor**

`clifford.tools.g3c.square_roots_of_rotor(R)`

Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition Leo Dorst and Robert Valkenburg

**clifford.tools.g3c.n\_th\_rotor\_root**

`clifford.tools.g3c.n_th_rotor_root(R, n)`

Takes the n\_th root of rotor R n must be a power of 2

**clifford.tools.g3c.interp\_objects\_root**

`clifford.tools.g3c.interp_objects_root(C1, C2, alpha)`

Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly linearly interpolates conformal objects Return a valid object from the addition result C

**clifford.tools.g3c.general\_object\_interpolation**

`clifford.tools.g3c.general_object_interpolation(object_alpha_array, object_list,  
new_alpha_array, kind='linear')`

Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 This is a general interpolation through the

**clifford.tools.g3c.average\_objects**

`clifford.tools.g3c.average_objects(obj_list, weights=[], check_grades=True)`

Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C

**clifford.tools.g3c.val\_average\_objects\_with\_weights**

`clifford.tools.g3c.val_average_objects_with_weights(obj_array, weights_array)`

Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C

**clifford.tools.g3c.val\_average\_objects**

`clifford.tools.g3c.val_average_objects(obj_array)`

Hadfield and Lasenby, Direct Linear Interpolation of Geometric Objects, AGACSE2018 Directly averages conformal objects Return a valid object from the addition result C

**clifford.tools.g3c.rotor\_between\_objects**

`clifford.tools.g3c.rotor_between_objects(X1, X2)`

Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a rotor that takes X1 to X2 Return a valid object from the addition result  $1 + \gamma X_2 X_1$

**clifford.tools.g3c.val\_rotor\_between\_objects\_root**

`clifford.tools.g3c.val_rotor_between_objects_root(X1, X2)`

**clifford.tools.g3c.val\_rotor\_between\_objects\_explicit**

`clifford.tools.g3c.val_rotor_between_objects_explicit(X1, X2)`

Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a rotor that takes X1 to X2

Implements an optimised version of:

```
gamma1 = (X1 * X1)[0]
gamma2 = (X2 * X2)[0]

M12 = X1*X2 + X2*X1
K = 2 + gamma1*M12

if np.sum(np.abs(K.value)) < 0.0000001:
    return 1 + 0*e1

if sum(np.abs(M12(4).value)) > 0.0000001:
    lamb = -(K(4) * K(4))[0]
    mu = K[0]**2 + lamb
    root_mu = np.sqrt(mu)
    if abs(lamb) < 0.0000001:
        beta = 1.0/(2*np.sqrt(K[0]))
    else:
        beta_sqrd = 1/(2*(root_mu + K[0]))
        beta = np.sqrt(beta_sqrd)
    R = (beta*K(4) - (1/(2*beta))) * (1 + gamma1*X2*X1)/(root_mu)
    return R
else:
    return (1 + gamma1*X2*X1)/(np.sqrt(abs(K[0])))
```

**clifford.tools.g3c.calculate\_S\_over\_mu**

`clifford.tools.g3c.calculate_S_over_mu(X1, X2)`

Lasenby and Hadfield AGACSE2018 For any two conformal objects X1 and X2 this returns a factor that corrects the X1 + X2 back to a blade

**clifford.tools.g3c.val\_rotor\_between\_lines**

`clifford.tools.g3c.val_rotor_between_lines(L1_val, L2_val)`

Implements a very optimised rotor line to line extraction

**clifford.tools.g3c.rotor\_between\_lines**

`clifford.tools.g3c.rotor_between_lines(L1, L2)`

Implements a very optimised rotor line to line extraction

**clifford.tools.g3c.rotor\_between\_planes**

`clifford.tools.g3c.rotor_between_planes(P1, P2)`

return the rotor between two planes

**clifford.tools.g3c.val\_rotor\_rotor\_between\_planes**

`clifford.tools.g3c.val_rotor_rotor_between_planes(P1_val, P2_val)`

**Submodules**

<i>object_fitting</i>	Tools for fitting geometric primitives to point clouds
-----------------------	--

**clifford.tools.g3c.object\_fitting**

Tools for fitting geometric primitives to point clouds

**Object Fitting**

<i>fit_circle</i> (point_list)	Performs Leo Dorsts circle fitting technique
<i>val_fit_circle</i> (point_list)	Performs Leo Dorsts circle fitting technique
<i>fit_line</i> (point_list)	Does line fitting with combo J.Lasenbys method and L.
<i>val_fit_line</i> (point_list)	Does line fitting with combo J.Lasenbys method and L.
<i>fit_sphere</i> (point_list)	Performs Leo Dorsts sphere fitting technique
<i>val_fit_sphere</i> (point_list)	Performs Leo Dorsts sphere fitting technique
<i>fit_plane</i> (point_list)	Does plane fitting with combo J.Lasenbys method and L.

continues on next page

Table 29 – continued from previous page

<code>val_fit_plane(point_list)</code>	Does plane fitting with combo J.Lasenbys method and L.
--	--

---

### **clifford.tools.g3c.object\_fitting.fit\_circle**

`clifford.tools.g3c.object_fitting.fit_circle(point_list)`  
 Performs Leo Dorsts circle fitting technique

### **clifford.tools.g3c.object\_fitting.val\_fit\_circle**

`clifford.tools.g3c.object_fitting.val_fit_circle(point_list)`  
 Performs Leo Dorsts circle fitting technique

### **clifford.tools.g3c.object\_fitting.fit\_line**

`clifford.tools.g3c.object_fitting.fit_line(point_list)`  
 Does line fitting with combo J.Lasenbys method and L. Dorsts

### **clifford.tools.g3c.object\_fitting.val\_fit\_line**

`clifford.tools.g3c.object_fitting.val_fit_line(point_list)`  
 Does line fitting with combo J.Lasenbys method and L. Dorsts

### **clifford.tools.g3c.object\_fitting.fit\_sphere**

`clifford.tools.g3c.object_fitting.fit_sphere(point_list)`  
 Performs Leo Dorsts sphere fitting technique

### **clifford.tools.g3c.object\_fitting.val\_fit\_sphere**

`clifford.tools.g3c.object_fitting.val_fit_sphere(point_list)`  
 Performs Leo Dorsts sphere fitting technique

### **clifford.tools.g3c.object\_fitting.fit\_plane**

`clifford.tools.g3c.object_fitting.fit_plane(point_list)`  
 Does plane fitting with combo J.Lasenbys method and L. Dorsts

**clifford.tools.g3c.object\_fitting.val\_fit\_plane**

`clifford.tools.g3c.object_fitting.val_fit_plane` (*point\_list*)

Does plane fitting with combo J.Lasenbys method and L. Dorsts

**2.3.2 Classifying conformal GAs**


---

*classify*

Tools for interpreting conformal blades

---

**clifford.tools.classify**

Tools for interpreting conformal blades

`clifford.tools.classify.classify` (*x*) → *clifford.tools.classify.Blade*

Classify a conformal multivector into a parameterized geometric description.

The multivector should be from a *ConformalLayout*, such as the one returned by *clifford.conformalize* ().

Implemented based on the approach described in table 14.1 of *Geometric Algebra for Computer Science (Revised Edition)*.

Example usage:

```
>>> from clifford.g3c import *

>>> classify(e1)
DualFlat[1] (flat=Plane(direction=-(1.0^e23), location=0))

>>> classify(einf)
InfinitePoint(direction=1.0)

>>> classify(up(e1))
Point(direction=1.0, location=(1.0^e1))

>>> classify(up(3*e1)^up(4*e2))
PointPair(direction=-(3.0^e1) + (4.0^e2), location=(1.5^e1) + (2.0^e2), radius=2.
↪5)

>>> classify(up(e1)^up(e2)^up(e1+2*e2))
Circle(direction=-(2.0^e12), location=(1.0^e1) + (1.0^e2), radius=1.0)

>>> classify(up(e1)^up(e2)^up(e1+2*e2)^einf)
Plane(direction=-(2.0^e12), location=0)

>>> classify(up(e1)^e2)
Tangent[2] (direction=(1.0^e2), location=(1.0^e1))

# how the inheritance works
>>> Point.mro()
[Point, Tangent[1], Round[1], Blade[1], Tangent, Round, Blade, object]
```

The reverse of this operation is *Blade.mv*.

The return type is a *Blade*:

**class** clifford.tools.classify.**Blade**(\*args, \*\*kwargs)

Base class for providing interpretation of blades.

Note that thanks to the unual metaclass, this class and its subclasses are have grade-specific specializations, eg `Blade[2]` is a type for blades of grade 2.

**layout**

The layout to which this blade belongs

**Type** *Layout*

**property** **mv**

Convert this back into its GA representation

The subclasses below are the four categories to which all blades belong, where  $E$  is a euclidean blade, and  $T_p[X]$  represents a translation of the conformal blade  $X$  by the euclidean vector  $p$ .

**class** clifford.tools.classify.**Direction**(\*args, \*\*kwargs)

$En_\infty$

**direction**

The euclidean direction,  $E$

**Type** *MultiVector*

**class** clifford.tools.classify.**Flat**(\*args, \*\*kwargs)

$T_p[n_o \wedge (En_\infty)]$

**direction**

The euclidean direction,  $E$

**Type** *MultiVector*

**location**

The closest point on this flat to the origin,  $p$ , as a euclidean vector.

**Type** *MultiVector*

**class** clifford.tools.classify.**DualFlat**(\*args, \*\*kwargs)

Dual of *Flat*

**flat**

The flat this is the dual of

**Type** *Flat*

**class** clifford.tools.classify.**Round**(\*args, \*\*kwargs)

$T_p[(n_o + \frac{1}{2}\rho^2 n_\infty)E]$

**direction**

The euclidean direction,  $E$

**Type** *MultiVector*

**location**

The euclidean center,  $p$

**Type** *MultiVector*

**radius**

The radius,  $\rho$ , which may be imaginary

**Type** *float* or *complex*

These can be constructed directly, and will attempt to show a grade-specific interpretation:



```
>>> from clifford.g3c import *
>>> Round(location=e1, direction=e1^e2, radius=1)
Circle(direction=(1^e12), location=(1^e1), radius=1)
>>> Round(direction=e1^e2^e3, location=e1, radius=1)
Sphere(direction=(1^e123), location=(1^e1), radius=1)
```

## Aliased types

In addition, aliases are created for specific grades of the above types, with more convenient names:

```
class clifford.tools.classify.Tangent(*args, **kwargs)
    Bases: clifford.tools.classify.Round

    A Round of radius 0,  $T_p[n_o E]$ 

class clifford.tools.classify.Point(*args, **kwargs)
    Bases: clifford.tools.classify.Tangent[1]

    A conformal point,  $A$ 

class clifford.tools.classify.PointFlat(*args, **kwargs)
    Bases: clifford.tools.classify.Flat[2]

    A flat point,  $A \wedge n_\infty$ 

class clifford.tools.classify.Line(*args, **kwargs)
    Bases: clifford.tools.classify.Flat[3]

    A line,  $A \wedge B \wedge n_\infty$ 

class clifford.tools.classify.Plane(*args, **kwargs)
    Bases: clifford.tools.classify.Flat[4]

    A line,  $A \wedge B \wedge C \wedge n_\infty$ 

class clifford.tools.classify.PointPair(*args, **kwargs)
    Bases: clifford.tools.classify.Round[2]

    A point pair,  $A \wedge B$ 

class clifford.tools.classify.Circle(*args, **kwargs)
    Bases: clifford.tools.classify.Round[3]

    A circle,  $A \wedge B \wedge C$ 

class clifford.tools.classify.Sphere(*args, **kwargs)
    Bases: clifford.tools.classify.Round[4]

    A sphere,  $A \wedge B \wedge C \wedge D$ 

class clifford.tools.classify.InfinitePoint(*args, **kwargs)
    Bases: clifford.tools.classify.Direction[1]

    A scalar multiple of  $n_\infty$ 
```

### 2.3.3 Determining Rotors From Frame Pairs or Orthogonal Matrices

Given two frames that are related by a orthogonal transform, we seek a rotor which enacts the transform. Details of the mathematics and pseudo-code used to create the algorithms below can be found at Allan Cortzen’s website, [Cor17].

There are also some helper functions which can be used to translate matrices into GA frames, so an orthogonal (or complex unitary) matrix can be directly translated into a Versor.

<code>orthoFrames2Versor(B[, A, delta, eps, det, ...])</code>	Determines versor for two frames related by an orthogonal transform
<code>orthoMat2Versor(A[, eps, I, is_complex])</code>	Translates an orthogonal (or unitary) matrix to a Versor
<code>mat2Frame(A[, I, is_complex])</code>	Translates a (possibly complex) matrix into a real vector frame
<code>frame2Mat(B[, A, I, is_complex])</code>	convert a list of vectors to a matrix
<code>func2Mat(f, I)</code>	Convert a function to a matrix by acting on standard basis
<code>omoh(A, B)</code>	Determines homogenization scaling for two <i>Frames</i> related by a Rotor
<code>rotor_decomp(V, x)</code>	Rotor decomposition of rotor V

#### clifford.tools.orthoFrames2Versor

`clifford.tools.orthoFrames2Versor(B, A=None, delta: float = 0.001, eps: Optional[float] = None, det=None, remove_scaling: bool = False)`

Determines versor for two frames related by an orthogonal transform

Based on [Cor17][FD11]. This works in Euclidean spaces and, under special circumstances in other signatures. See [Cor17] for limitations/details.

##### Parameters

- **B** (*list of vectors, or clifford.Frame*) – the set of vectors after the transform, and homogenization. ie  $B = (B/B|einf)$
- **A** (*list of vectors, or clifford.Frame*) – the set of vectors before the transform. If *None* we assume A is the basis given by `B.layout.basis_vectors_lst`.
- **delta** (*float*) – Tolerance for reflection/rotation determination. If the normalized distance between `A[i]` and `B[i]` is larger than delta, we use reflection, otherwise use rotation.
- **eps** (*float*) – Tolerance on spinor determination. if pseudoscalar of A differs in magnitude from pseudoscalar of B by eps, then we have spinor. If *None*, use the `clifford.eps()` global eps.
- **det** (*[+1, -1, None]*) – The sign of the determinant of the versor, if known. If it is known a-priori that the versor is a rotation vs a reflection, this fact might be needed to correctly append an additional reflection which leaves transformed points invariant. See 4.6.3 of [FD11].
- **remove\_scaling** (*bool*) – Remove the effects of homogenization from frame B. This is needed if you are working in CGA, but the input data is given in the original space. See `omoh()` for more. See 4.6.2 of [FD11].

##### Returns

- **R** (*clifford.MultiVector*) – the Versor.
- **rs** (*list of clifford.MultiVector*) – ordered list of found reflectors/rotors.

### clifford.tools.orthoMat2Versor

`clifford.tools.orthoMat2Versor` (*A*, *eps=None*, *I=None*, *is\_complex=None*)

Translates an orthogonal (or unitary) matrix to a Versor

*A* is interpreted as the frame produced by transforming a orthonormal frame by an orthogonal transform. Given this relation, this function will find the versor which enacts this transform.

#### Parameters

- **A** (*matrix*) – matrix to be transformed
- **eps** (*number*) – tolerance
- **I** (*MultiVector*) – GA of A
- **is\_complex** (*boolean*) – is A complex?

### clifford.tools.mat2Frame

`clifford.tools.mat2Frame` (*A*: *numpy.ndarray*, *I*: *Optional[clifford.\_multivector.MultiVector]*  
                               = *None*, *is\_complex*: *Optional[bool]* = *None*)  
                               → *Tuple[List[clifford.\_multivector.MultiVector], clifford.\_multivector.MultiVector]*

Translates a (possibly complex) matrix into a real vector frame

The rows and columns of *A* are interpreted as follows

- *M*, *N* = *A*.shape
- *M*: dimension of space
- *N*: number of vectors

If *A* is complex *M* and *N* are doubled.

#### Parameters

- **A** (*ndarray*) – *M*×*N* matrix representing vectors
- **I** (*MultiVector*) – if none we generate an algebra of *G<sub>N</sub>*, if layout we take the vector basis from that, and if its a list we will assume its a vector basis.

#### Returns

- **a** (*list of clifford.MultiVector*) – The resulting vectors
- **I** (*clifford.MultiVector*) – The blade holding the vectors in *a*.

### clifford.tools.frame2Mat

`clifford.tools.frame2Mat` (*B*, *A=None*, *I=None*, *is\_complex=None*)

convert a list of vectors to a matrix

#### Parameters

- **B** (*list*) – a list of vectors that have been transformed
- **A** (*None*, *list of vectors*) – a list of vectors in their initial state. if none we assume orthonormal basis given by *B.pseudoScalar*, or *I*
- **I** (*MultiVector*, *None*) – pseudoscalar of the space. if *None*, we use *B.pseudoScalar*
- **is\_complex** (*Bool*) – do you want a complex matrix?

### clifford.tools.func2Mat

`clifford.tools.func2Mat(f, I)`

Convert a function to a matrix by acting on standard basis

#### Parameters

- **f** (*function*) – function that maps vectors to vectors
- **I** (`MultiVector`) – psuedoscalar of basis

See also:

`frame2Mat`

### clifford.tools.omoh

`clifford.tools.omoh(A: Union[clifford._frame.Frame, List[clifford._multivector.MultiVector]], B: Union[clifford._frame.Frame, List[clifford._multivector.MultiVector]]) → numpy.ndarray`

Determines homogenization scaling for two *Frames* related by a Rotor

This is used as part of the `orthoFrames2Versor()` algorithm, when the frames are given in CGA. It is required because the model assumes,  $B = R * A * \sim R$ , but if data is given in the original space, only  $\lambda * B$  '== homo(B) is observable.

We need to determine lambda before the Cartan-based algorithm can be used. The name of this function is the reverse of `homo()`, which is the method used to homogenize.

#### Parameters

- **A** (*list of vectors, or clifford.Frame*) – the set of vectors before the transform
- **B** (*list of vectors, or clifford.Frame*) – the set of vectors after the transform, and homogenization. ie  $B = B / (B | e_{inf})$

**Returns out** – weights on *B*, which produce inhomogenous versions of *B*. If you multiply the input *B* by *lam*, it will fulfill  $B = R * A * \sim R$

**Return type** array of floats

### Examples

```
>>> lam = ohom(A, B)
>>> B_ohom = Frame([B[k]*lam[k] for k in range(len(B))])
```

### clifford.tools.rotor\_decomp

`clifford.tools.rotor_decomp(V: clifford._multivector.MultiVector, x: clifford._multivector.MultiVector) → Tuple[clifford._multivector.MultiVector, clifford._multivector.MultiVector]`

Rotor decomposition of rotor V

Given a rotor V, and a vector x, this will decompose V into a series of two rotations, U and H, where U leaves x invariant and H contains x.

Limited to 4D for now.

See [Hes15], Appendix B, Theorem 4.

#### Parameters

- $\mathbf{V}$  (`clifford.MultiVector`) – rotor
- $\mathbf{x}$  (`clifford.MultiVector`) – vector

#### Returns

- $\mathbf{H}$  (`clifford.Multivector`) – rotor which contains  $\mathbf{x}$
- $\mathbf{U}$  (`clifford.Multivector`) – rotor which leaves  $\mathbf{x}$  invariant

## 2.4 operator functions (`clifford.operator`)

This module exists to enable functional programming via `functools.reduce()`. It can be thought of as equivalent to the builtin `operator` module, but for the operators from geometric algebra.

```
>>> import functools
>>> import clifford.operator
>>> from clifford.g3 import *
>>> Ms = [e1, e1 + e2, e2 + e3] # list of multivectors
>>> assert functools.reduce(clifford.operator.op, Ms) == Ms[0] ^ Ms[1] ^ Ms[2]
```

`clifford.operator.gp` ( $M, N$ )  
Geometric product function  $MN$ , equivalent to  $M * N$ .

$M$  and  $N$  must be from the same layout

`clifford.operator.op` ( $M, N$ )  
Outer product function  $M \wedge N$ , equivalent to  $M ^ N$ .

$M$  and  $N$  must be from the same layout

`clifford.operator.ip` ( $M, N$ )  
Hestenes inner product function  $M \bullet N$ , equivalent to  $M | N$ .

$M$  and  $N$  must be from the same layout

Changed in version 1.3.0: These functions used to be in `clifford`, but have been moved to this submodule.

## 2.5 transformations (`clifford.transformations`)

New in version 1.3.0.

This module may in future become the home for optimized rotor transformations, or non-linear transformations.

See the [Linear transformations](#) tutorial for an introduction to how to use parts of this module.

## 2.5.1 Base classes

`clifford.transformations.Transformation`

A callable mapping one MultiVector to another.

alias of `Callable[[clifford._multivector.MultiVector], clifford._multivector.MultiVector]`

**class** `clifford.transformations.Linear` (*\*args, \*\*kws*)

A transformation which is linear, such that for scalar  $a_i$ ,  $f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2)$ .

**class** `clifford.transformations.FixedLayout` (*layout\_src: clifford.\_layout.Layout, layout\_dst: Optional[clifford.\_layout.Layout] = None*)

A transformation with a fixed source and destination layout.

### Parameters

- **layout\_src** (*Layout* of  $S$  dimensions) – The layout from which this transformation takes multivectors as input
- **layout\_dst** (*Layout* of  $D$  dimensions) – The layout in which this transformation produces multivectors as output. Defaults to the same as the input.

## 2.5.2 Matrix-backed implementations

**class** `clifford.transformations.LinearMatrix` (*matrix, layout\_src: clifford.\_layout.Layout, layout\_dst: Optional[clifford.\_layout.Layout] = None*)

Linear transformation implemented by a matrix

Transformations need not be grade preserving.

### Parameters

- **matrix** (*(2\*\*D, 2\*\*S) array\_like*) – A matrix that transforms multivectors from *layout\_src* with  $2^S$  elements to multivectors in *layout\_dst* with  $2^D$  elements, by left-multiplication.
- **layout\_src** (*Layout of S dimensions*) – Passed on to *FixedLayout*.
- **layout\_dst** (*Layout of D dimensions*) – Passed on to *FixedLayout*.

See also:

*clifford.BladeMap* A faster but less general approach that works on basis blades

### property adjoint

The adjoint transformation

**classmethod from\_function** (*func: Callable[[clifford.\_multivector.MultiVector], clifford.\_multivector.MultiVector], layout\_src: clifford.\_layout.Layout, layout\_dst: Optional[clifford.\_layout.Layout] = None*)  $\rightarrow$  *clifford.transformations.LinearMatrix*

Build a linear transformation from the result of a function applied to each basis blade.

### Parameters

- **func** – A function taking basis blades from *layout\_src* that produces multivectors in *layout\_dst*.
- **layout\_src** (*Layout of S dimensions*) – The layout to pass into the generating function

- **layout\_dst** (*Layout of D dimensions*) – The layout the generating function is expected to produce. If not passed, this is inferred.

### Example

```
>>> from clifford import transformations, Layout
>>> l = Layout([1, 1])
>>> e1, e2 = l.basis_vectors_lst
>>> rot_90 = transformations.LinearMatrix.from_function(lambda x: (1 +
↳ e1*e2)*x*(1 - e1*e2)/2, l)
>>> rot_90(e1)
(1.0^e2)
>>> rot_90(e2)
-(1.0^e1)
>>> rot_90(e1*e2)
(1.0^e12)
```

See also:

**LinearMatrix.from\_rotor** a shorter way to spell the above example

**clifford.linear\_operator\_as\_matrix** a lower-level function for working with a subset of basis blades

**classmethod from\_rotor** (*rotor:* *clifford.\_multivector.MultiVector*) → *clifford.transformations.LinearMatrix*

Build a linear transformation from the result of applying a rotor sandwich.

The resulting transformation operates within the algebra of the provided rotor.

**Parameters rotor** – The rotor to apply

### Example

```
>>> from clifford import transformations, Layout
>>> l = Layout([1, 1])
>>> e1, e2 = l.basis_vectors_lst
>>> rot_90 = transformations.LinearMatrix.from_rotor(1 + e1*e2)
>>> rot_90(e1)
(1.0^e2)
>>> rot_90(e2)
-(1.0^e1)
>>> rot_90(e1*e2)
(1.0^e12)
```

**class clifford.transformations.OutermorphismMatrix** (*matrix*, *layout\_src:* *clifford.\_layout.Layout*, *layout\_dst:* *Optional[clifford.\_layout.Layout]* = *None*)

A generalization of a linear transformation to vectors via the outer product.

Namely, given a linear transformation  $F(u) \rightarrow v$ , this generalizes to the blades by outermorphism,  $F(u_1 \wedge u_2) \rightarrow F(u_1) \wedge F(u_2)$ , and to the multivectors by distributivity.

Such a transformation is grade preserving.

See [DFM09] Chapter 4 for more information

**Parameters**

- **matrix** ( $(D, S)$  *array\_like*) – A matrix that transforms vectors from *layout\_src* of size *S* to vectors in *layout\_dst* of size *D* by left-multiplication.
- **layout\_src** (*Layout of S dimensions*) – Passed on to *FixedLayout*.
- **layout\_dst** (*Layout of D dimensions*) – Passed on to *FixedLayout*.

**Example**

We can construct a simple transformation that permutes and non-uniformly scales the basis vectors:

```
>>> from clifford import transformations, Layout
>>> layout = Layout([1, 1, 1])
>>> e1, e2, e3 = layout.basis_vectors_lst
>>> layout_new = Layout([1, 1, 1], names='f')
>>> m = np.array([[0, 1, 0],
...               [0, 0, 2],
...               [3, 0, 0]])
>>> lt = transformations.OutermorphismMatrix(m, layout, layout_new)
```

Applying it to some multivectors:

```
>>> # the transformation we specified
>>> lt(e1), lt(e2), lt(e3)
((3^f3), (1^f1), (2^f2))

>>> # the one deduced by outermorphism
>>> lt(e1^e2), lt(e2^e3), lt(e1^e3)
(-(3^f13), (2^f12), -(6^f23))

>>> # and by distributivity
>>> lt(1 + (e1^e2^e3))
1 + (6^f123)
```

**2.5.3 Helper functions**

`clifford.transformations.between_basis_vectors` (*layout\_src*: *clifford.\_layout.Layout*,  
*layout\_dst*: *clifford.\_layout.Layout*,  
*mapping*: *Optional[Dict[Any, Any]]* = *None*) → *clifford.transformations.OutermorphismMatrix*

Construct an outermorphism that maps basis vectors from one layout to basis vectors in another.

**Parameters**

- **layout\_src** (*Layout*) – Passed on to *FixedLayout*.
- **layout\_dst** (*Layout*) – Passed on to *FixedLayout*.
- **mapping** – If provided, a dictionary mapping the ids of source basis vectors to the ids of destination basis vectors. For example, {1: 2, 2: 3, 3: 1} would permute the basis vectors of *clifford.g3*.



**Example**

See the tutorial on *Working with custom algebras* for a motivating example.



## PREDEFINED ALGEBRAS

The easiest way to get started with `clifford` is to use one of several predefined algebras:

- `g2`: 2D Euclidean, `Cl(2)`. See [Quick Start \(G2\)](#) for some examples.
- `g3`: 3D Euclidean, `Cl(3)`. See [The Algebra Of Space \(G3\)](#) for some examples.
- `g4`: 4D Euclidean, `Cl(4)`.
- `g2c`: Conformal space for G2, `Cl(3, 1)`. See [Conformal Geometric Algebra](#) for some examples.
- `g3c`: Conformal space for G3, `Cl(4, 1)`.
- `pga`: Projective space for G3 `Cl(3, 0, 1)`.
- `gac`: Geometric Algebra for Conics, `Cl(5, 3)`.
- `dpga`: Double PGA also referred to as the Mother Algebra, `Cl(4, 4)`.
- `dg3c`: Double Conformal Geometric Algebra, effectively two `g3c` algebras glued together `Cl(8, 2)`.

By using the pre-defined algebras in place of calling `Cl` directly, you will often find that your program starts up faster.

**`clifford.<predefined>.e<ijk>`**

All of these modules expose the basis blades as attributes, and can be used like so

```
In [1]: from clifford import g2
In [2]: g2.e1 * g2.e2
Out[2]: (1^e12)
```

Additionally, they define the following attributes, which contain the return values of `clifford.Cl()`:

**`clifford.<predefined>.layout`**

The associated `clifford.Layout`

```
In [3]: g2.layout
Out[3]:
Layout([1, 1],
       ids=BasisVectorIds.ordered_integers(2),
       order=BasisBladeOrder.shortlex(2),
       names=['', 'e1', 'e2', 'e12'])
```

**`clifford.<predefined>.blades`**

A shorthand for `Layout.blades()`

```
In [4]: g2.blades
Out[4]: {'': 1, 'e1': (1^e1), 'e2': (1^e2), 'e12': (1^e12)}
```

For interactive use, it's very handy to use `import *`

```
In [5]: from clifford.g2 import *  
  
In [6]: e1, e2, e12  
Out[6]: ((1^e1), (1^e2), (1^e12))
```

For the conformal layouts *g2c* and *g3c*, the full contents of the `stuff` result of `clifford.conformalize()` is also exposed as members of the module.

## CHANGELOG

### 4.1 Changes in 1.4.x

- Projection using `Multivector.__call__()` no longer raises `ValueError` for grades not present in the algebra, and instead just returns zero.
- Where possible, `Multivectors` preserve their data type in the dual, and the right and left complements.

### 4.2 Changes in 1.3.x

- Python 3.8 is officially supported. 1.2.0 was pinned to a bad numba version that was incompatible with 3.8.
- A new `clifford.operator` module to contain the previously undocumented `gp()`, `op()`, and `ip()` helpers.
- A new `clifford.transformations` module for linear transformations.
- Two new *Predefined Algebras*, `clifford.dpga` and `clifford.dg3c`.
- Improvements throughout the documentation:
  - Better overall structure, visible in the docs sidebar.
  - New tutorials for *Conformal Geometric Algebra* on visualization and applications.
  - New tutorial on *Working with custom algebras*.
  - New tutorial on *Linear transformations*.
  - New links at the top of each notebook tutorial, to run examples from the browser.
- Faster algebra construction. `C1(3)` is now 100× faster, and `C1(6)` is 20× faster. This is achieved by deferring product JIT compilation until the product is used for the first time.
- Additional testing and assorted improvements for `clifford.tools.g3c`:
  - `closest_point_on_circle_from_line()` has now been implemented roughly following the procedure described in Appendix A of [Andreas Aristidou's PhD thesis](#).
  - `closest_point_on_line_from_circle()` has now also been added, projecting the result of `closest_point_on_circle_from_line()` to the line.
- `clifford.ugly()` now results in less ugly output for *Predefined Algebras*.

### 4.2.1 Bugs fixed

- `MultiVector.meet()` no longer produces zero erroneously.
- `mv[e1 + e12]` now raises `ValueError`, rather than being interpreted as `mv[e1]`.
- `ip()` (the inner product) no longer performs the outer product.
- `Layout.parse_multivector()` now throws `SyntaxError` on invalid input, rather than silently producing nonsense.
- `Layout.parse_multivector()` supports basis vector names which do not start with e.
- In `clifford.tools.g3c`:
  - `val_midpoint_between_lines()` now handles the case that the two lines are touching.
  - `val_fit_circle()` now correctly selects the first and second eigenvalue regardless of order.
  - `sphere_beyond_plane()` now tested and correct.
  - `sphere_behind_plane()` now tested and correct.
  - `val_unsign_sphere()` is now jitted, as it should have been from the start.
  - `get_nearest_plane_point()` correctly returns the conformal point rather than the 3D point.

### 4.2.2 Compatibility notes

- `clifford.grades_present` is deprecated in favor of `MultiVector.grades()`, the latter of which now takes an `eps` argument.
- `del mv[i]` is no longer legal, the equivalent `mv[i] = 0` should be used instead.
- `Layout.dict_to_multivector` has been removed. It was accidentally broken in 1.0.5, so there is little point deprecating it.
- `Layout.basis_names()` now returns a list of `str`, rather than a numpy array of bytes. The result now matches the construction order, rather than being sorted alphabetically. The order of `Layout.metric()` has been adjusted for consistency.
- The `imt_prod_mask`, `omt_prod_mask`, and `lcmt_prod_mask` attributes of `Layout` objects have been removed, as these were an unnecessary intermediate computation that had no need to be public.
- Some functions in `clifford.tools.g3c` have been renamed:
  - `closest_points_on_circles` has been renamed to `iterative_closest_points_on_circles()`.
  - `closest_points_circle_line` has been renamed to `iterative_closest_points_circle_line()`.
  - `furthest_points_on_circles` has been renamed to `iterative_furthest_points_on_circles()`.
- While this release is compatible with numba version 0.49.0, it is recommended to use 0.48.0 which does not emit as many warnings. See the [Installation](#) instructions for how to follow this guidance.

### 4.2.3 Patch releases

- 1.3.1: Added compatibility with numba version 0.50.0.

## 4.3 Changes in 1.2.x

- `layout.isconformal`, `layout.einf`, and `layout.eo`, which were added in 1.0.4, have been removed. The first can now be spelt `isinstance(layout, clifford.ConformalLayout)`, and the other properties now exist only on `ConformalLayout` objects.
- `MultiVector.left_complement()` has been added for consistency with `MultiVector.right_complement()`.
- A new `clifford.tools.classify` module has been added for classifying blades.
- `Layout` objects print slightly more cleanly in Jupyter notebooks.
- `Layout.scalar` is now integral rather than floating point

### 4.3.1 Bugs fixed

- `pow(mv, 0)` gives the right result
- `nan` is now printed correctly when it appears in multivectors. Previously it was hidden
- `MultiVector.right_complement()` no longer performs the left complement.
- `MultiVector.vee()` has been corrected to have the same sign as `MultiVector.meet()`

### 4.3.2 Compatibility notes

- `Layout.scalar` is now integral rather than floating point, to match `Layout.pseudoScalar`.

## 4.4 Changes in 1.1.x

- Restores `layout.gmt`, `Layout.omt`, `Layout.imt`, and `Layout.lcmt`. A few releases ago, these existed but were dense. For memory reasons, they were then removed entirely. They have now been reinstated as `sparse.COO` matrix objects, which behave much the same as the original dense arrays.
- `MultiVectors` preserve their data type in addition, subtraction, and products. This means that integers remain integers until combined with floats. Note that this means in principle integer overflow is possible, so working with floats is still recommended. This also adds support for floating point types of other precision, such as `np.float32`.
- `setup.py` is now configured such that `pip2 install clifford` will not attempt to download this version, since it does not work at all on python 2.
- Documentation now includes examples of `pyganja` visualizations.

### 4.4.1 Compatibility notes

- `Layout.blades()` now includes the scalar 1, as do other similar functions.
- `MultiVector.grades()` now returns a `set` not a `list`. This means code like `mv.grades() == [0]` will need to change to `mv.grades() == {0}`, or to work both before and after this change, `set(mv.grades()) == {0}`.

### 4.4.2 Bugs fixed

- `mv[(i, j)]` would sometimes fail if the indices were not in canonical order.
- `mv == None` and `layout == None` would crash rather than return `False`.
- `blade.isVersor()` would return `False`.
- `layout.blades_of_grade(0)` would not return the list it claimed to return.

### 4.4.3 Internal changes

- Switch to `pytest` for testing.
- Enable code coverage.
- Split into smaller files.
- Remove python 2 compatibility code, which already no longer worked.

## 4.5 Changes 0.6-0.7

- Added a real license.
- Convert to `NumPy` instead of `Numeric`.

## 4.6 Changes 0.5-0.6

- `join()` and `meet()` actually work now, but have numerical accuracy problems
- added `clean()` to *MultiVector*
- added `leftInv()` and `rightInv()` to *MultiVector*
- moved `pseudoScalar()` and `invPS()` to *MultiVector* (so we can derive new classes from *MultiVector*)
- changed all of the instances of creating a new *MultiVector* to create an instance of `self.__class__` for proper inheritance
- fixed bug in `laInv()`
- fixed the massive confusion about how `dot()` works
- added left-contraction
- fixed embarrassing bug in `gmt` generation
- added `normal()` and `anticommutator()` methods



- fixed dumb bug in `elements()` that limited it to 4 dimensions

## 4.7 Acknowledgements

Konrad Hinsien fixed a few bugs in the conversion to numpy and adding some unit tests.



## ISSUES

**Warning:** This document is kept for historic reasons, but may no longer reflect the current state of the latest release of `clifford`. For the most up to date source of issues, look at [the GitHub issue tracker](#).

- Currently, algebras over 6 dimensions are very slow. this is because this module was written for *pedagogical* purposes. However, because the syntax for this module is so attractive, we plan to fix the performance problems, in the future...
- Due to Python's [order of operations](#), the bit operators `^ << |` are evaluated after the normal arithmetic operators `+ - * /`, which do not follow the precedence expected in GA

# written	meaning	possibly intended
<code>1^e1 + 2^e2</code>	<code>== 1^(e1+2)^e2</code>	<code>!= (1^e0) + (2^e1)</code>
<code>e2 + e1 e2</code>	<code>== (e2 + e1) e2</code>	<code>!= e1 + (e1 e2)</code>

This can also cause confusion within the bitwise operators:

# written	meaning	possibly intended
<code>e1 &lt;&lt; e2 ^ e1</code>	<code>== (e1 &lt;&lt; e2) ^ e1</code>	<code>!= e1 &lt;&lt; (e2 ^ e1)</code>
<code>e1 ^ e2   e1</code>	<code>== (e1 &lt;&lt; e2) ^ e1</code>	<code>!= e1 &lt;&lt; (e2 ^ e1)</code>

- Since `|` is the inner product and the inner product with a scalar vanishes by definition, an expression like:

```
(1|e0) + (2|e1)
```

is null. Use the outer product or full geometric product, to multiply scalars with *MultiVectors*. This can cause problems if one has code that mixes Python numbers and *MultiVectors*. If the code multiplies two values that can each be either type without checking, one can run into problems as `1 | 2` has a very different result from the same multiplication with scalar *MultiVectors*.

- Taking the inverse of a *MultiVector* will use a method proposed by Christian Perwass that involves the solution of a matrix equation. A description of that method follows:

Representing multivectors as  $2^{\text{dims}}$ -vectors (in the matrix sense), we can carry out the geometric product with a multiplication table. In pseudo-tensorish language (using summation notation)

$$m_i g_{ijk} n_k = v_j$$

Suppose  $m_i$  are known ( $M$  is the vector we are taking the inverse of), the  $g_{ijk}$  have been computed for this algebra, and  $v_j = 1$  if the  $j$ 'th element is the scalar element and 0 otherwise, we can compute the dot product  $m_i g_{ijk}$ . This yields a rank-2 matrix. We can then use well-established computational linear algebra techniques to solve this matrix equation for  $n_k$ . The `laInv` method does precisely that.

The usual, analytic, method for computing inverses ( $M^{-1} = \tilde{M}/(M\tilde{M})$  iff  $M\tilde{M} = |M|^2$ ) fails for those multivectors where  $M \sim M$  is not a scalar. It is only used if the `inv` method is manually set to point to `normalInv`.

My testing suggests that `laInv` works. In the cases where `normalInv` works, `laInv` returns the same result (within `_eps`). In all cases, `M * M.laInv() == 1.0` (within `_eps`). Use whichever you feel comfortable with.

Of course, a new issue arises with this method. The inverses found are sometimes dependant on the order of multiplication. That is:

```
M.laInv() * M == 1.0
M * M.laInv() != 1.0
```

XXX Thus, there are two other methods defined, `leftInv` and `rightInv` which point to `leftLaInv` and `rightLaInv`. The method `inv` points to `rightInv`. Should the user choose, `leftInv` and `rightInv` will both point to `normalInv`, which yields a left- and right-inverse that are the same should either exist (the proof is fairly simple).

- The basis vectors of any algebra will be orthonormal unless you supply your own multiplication tables (which you are free to do after the `Layout` constructor is called). A derived class could be made to calculate these tables for you (and include methods for generating reciprocal bases and the like).
- No care is taken to preserve the dtype of the arrays. The purpose of this module is pedagogical. If your application requires so many multivectors that storage becomes important, the class structure here is unsuitable for you anyways. Instead, use the algorithms from this module and implement application-specific data structures.
- Conversely, explicit typecasting is rare. `MultiVectors` will have integer coefficients if you instantiate them that way. Dividing them by Python integers will have the same consequences as normal integer division. Public outcry will convince me to add the explicit casts if this becomes a problem.

---

Happy hacking!

Robert Kern

[robert.kern@gmail.com](mailto:robert.kern@gmail.com)

The following section was generated from `docs/tutorials/g2-quick-start.ipynb` .....

## QUICK START (G2)

This notebook gives a terse introduction to using the `clifford` module, using a two-dimensional geometric algebra as the context.

### 6.1 Setup

First, import `clifford` and instantiate a two-dimensional algebra (G2),

```
[1]: import clifford as cf

layout, blades = cf.Cl(2) # creates a 2-dimensional clifford algebra
```

Inspect blades.

```
[2]: blades
[2]: {'': 1, 'e1': (1^e1), 'e2': (1^e2), 'e12': (1^e12)}
```

Assign blades to variables

```
[3]: e1 = blades['e1']
     e2 = blades['e2']
     e12 = blades['e12']
```

### 6.2 Basics

```
[4]: e1*e2 # geometric product
[4]: (1^e12)
```

```
[5]: e1|e2 # inner product
[5]: 0
```

```
[6]: e1^e2 # outer product
[6]: (1^e12)
```

## 6.3 Reflection

```
[7]: a = e1+e2      # the vector
     n = e1        # the reflector
     -n*a*n.inv()  # reflect `a` in hyperplane normal to `n`

[7]: -(1.0^e1) + (1.0^e2)
```

## 6.4 Rotation

```
[8]: from math import e, pi

     R = e**(pi/4*e12) # enacts rotation by pi/2
     R

[8]: 0.70711 + (0.70711^e12)
```

```
[9]: R*e1*~R      # rotate e1 by pi/2 in the e12-plane

[9]: -(1.0^e2)
```

..... doc/tutorials/g2-quick-start.ipynb ends here.

The following section was generated from docs/tutorials/g3-algebra-of-space.ipynb .....

## THE ALGEBRA OF SPACE (G3)

In this notebook, we give a more detailed look at how to use `clifford`, using the algebra of three dimensional space as a context.

### 7.1 Setup

First, we import `clifford` as `cf`, and instantiate a three dimensional geometric algebra using `Cl()` ([docs](#)).

```
[1]: import clifford as cf

layout, blades = cf.Cl(3) # creates a 3-dimensional clifford algebra
```

Given a three dimensional GA with the orthonormal basis,

$$e_i \cdot e_j = \delta_{ij}$$

The basis consists of scalars, three vectors, three bivectors, and a trivector.

$$\left\{ \underbrace{\alpha}_{\text{scalar}}, \underbrace{e_1, e_2, e_3}_{\text{vectors}}, \underbrace{e_{12}, e_{23}, e_{13}}_{\text{bivectors}}, \underbrace{e_{123}}_{\text{trivector}} \right\}$$

`Cl()` creates the algebra and returns a `layout` and `blades`. The `layout` holds information and functions related this instance of G3, and the `blades` is a dictionary which contains the basis blades, indexed by their string representations,

```
[2]: blades
[2]: {'': 1,
      'e1': (1^e1),
      'e2': (1^e2),
      'e3': (1^e3),
      'e12': (1^e12),
      'e13': (1^e13),
      'e23': (1^e23),
      'e123': (1^e123)}
```

You may wish to explicitly assign the blades to variables like so,

```
[3]: e1 = blades['e1']
     e2 = blades['e2']
     # etc ...
```

Or, if you're lazy and just working in an interactive session you can use `locals()` to update your namespace with all of the blades at once.

```
[4]: locals().update(blades)
```

Now, all the blades have been defined in the local namespace

```
[5]: e3, e123
```

```
[5]: ((1^e3), (1^e123))
```

## 7.2 Basics

### 7.2.1 Products

The basic products are available

```
[6]: e1*e2 # geometric product
```

```
[6]: (1^e12)
```

```
[7]: e1|e2 # inner product
```

```
[7]: 0
```

```
[8]: e1^e2 # outer product
```

```
[8]: (1^e12)
```

```
[9]: e1^e2^e3 # even more outer products
```

```
[9]: (1^e123)
```

### 7.2.2 Defects in Precedence

Python's operator precedence makes the outer product evaluate after addition. This requires the use of parentheses when using outer products. For example

```
[10]: e1^e2 + e2^e3 # fail, evaluates as
```

```
[10]: (2^e123)
```

```
[11]: (e1^e2) + (e2^e3) # correct
```

```
[11]: (1^e12) + (1^e23)
```

Also the inner product of a scalar and a Multivector is 0,

```
[12]: 4|e1
```

```
[12]: 0
```

So for scalars, use the outer product or geometric product instead

```
[13]: 4*e1
```

```
[13]: (4^e1)
```



### 7.2.3 Multivectors

Multivectors can be defined in terms of the basis blades. For example you can construct a rotor as a sum of a scalar and bivector, like so

```
[14]: import math

theta = math.pi/4
R = math.cos(theta) - math.sin(theta)*e23
R
[14]: 0.70711 - (0.70711^e23)
```

You can also mix grades without any reason

```
[15]: A = 1 + 2*e1 + 3*e12 + 4*e123
A
[15]: 1 + (2^e1) + (3^e12) + (4^e123)
```

### 7.2.4 Reversion

The reversion operator is accomplished with the tilde ~ in front of the Multivector on which it acts

```
[16]: ~A
[16]: 1 + (2^e1) - (3^e12) - (4^e123)
```

### 7.2.5 Grade Projection

Taking a projection onto a specific grade  $n$  of a Multivector is usually written

$$\langle A \rangle_n$$

can be done by using soft brackets, like so

```
[17]: A(0)  # get grade-0 elements of R
[17]: 1

[18]: A(1)  # get grade-1 elements of R
[18]: (2^e1)

[19]: A(2)  # you get it
[19]: (3^e12)
```

## 7.2.6 Magnitude

Using the reversion and grade projection operators, we can define the magnitude of  $A$

$$|A|^2 = \langle A\tilde{A} \rangle$$

```
[20]: (A*~A) (0)
```

```
[20]: 30
```

This is done in the `abs()` operator

```
[21]: abs(A)**2
```

```
[21]: 30.0
```

## 7.2.7 Inverse

The inverse of a Multivector is defined as  $A^{-1}A = 1$

```
[22]: A.inv()*A
```

```
[22]: 1.0
```

```
[23]: A.inv()
```

```
[23]: 0.13415 + (0.12195^e1) - (0.14634^e3) + (0.18293^e12) + (0.09756^e23) - (0.29268^e123)
```

## 7.2.8 Dual

The dual of a multivector  $A$  can be defined as

$$AI^{-1}$$

Where,  $I$  is the pseudoscalar for the GA. In  $G_3$ , the dual of a vector is a bivector,

```
[24]: a = 1*e1 + 2*e2 + 3*e3
a.dual()
```

```
[24]: -(3^e12) + (2^e13) - (1^e23)
```

## 7.2.9 Pretty, Ugly, and Display Precision

You can toggle pretty printing with `pretty()` or `ugly()`. `ugly()` returns an eval-able string.

```
[25]: cf.ugly()
A.inv()
```

```
[25]: MultiVector(Layout([1, 1, 1],
                        ids=BasisVectorIds.ordered_integers(3),
                        order=BasisBladeOrder.shortlex(3),
                        names=['', 'e1', 'e2', 'e3', 'e12', 'e13', 'e23', 'e123']),
                [0.13414634146341464, 0.12195121951219512, 0.0, -0.14634146341463414, 0.
↪ 18292682926829268, 0.0, 0.0975609756097561, -0.2926829268292683])
```

You can also change the displayed precision

```
[26]: cf.pretty(precision=2)
A.inv()
[26]: 0.13 + (0.12^e1) - (0.15^e3) + (0.18^e12) + (0.1^e23) - (0.29^e123)
```

This does not effect the internal precision used for computations.

## 7.3 Applications

### 7.3.1 Reflections

Reflecting a vector  $c$  about a normalized vector  $n$  is pretty simple,

$$c \rightarrow nc n$$

```
[27]: c = e1+e2+e3      # a vector
      n = e1            # the reflector
      n*c*n            # reflect 'a' in hyperplane normal to 'n'
[27]: (1^e1) - (1^e2) - (1^e3)
```

Because we have the `inv()` available, we can equally well reflect in un-normalized vectors using,

$$a \rightarrow nan^{-1}$$

```
[28]: a = e1+e2+e3      # the vector
      n = 3*e1          # the reflector
      n*a*n.inv()
[28]: (1.0^e1) - (1.0^e2) - (1.0^e3)
```

Reflections can also be made with respect to the a 'hyperplane normal to the vector  $n$ ', in this case the formula is negated

$$c \rightarrow -nc n^{-1}$$

### 7.3.2 Rotations

A vector can be rotated using the formula

$$a \rightarrow Ra\tilde{R}$$

Where  $R$  is a rotor. A rotor can be defined by multiple reflections,

$$R = mn$$

or by a plane and an angle,

$$R = e^{-\frac{\theta}{2}\hat{B}}$$

For example

```
[29]: R = math.e**(-math.pi/4*e12) # enacts rotation by pi/2
      R
[29]: 0.71 - (0.71^e12)

[30]: R*e1~R # rotate e1 by pi/2 in the e12-plane
[30]: (1.0^e2)
```

### 7.3.3 Some Ways to use Functions

Maybe we want to define a function which can return rotor of some angle  $\theta$  in the  $e_{12}$ -plane,

$$R_{12} = e^{-\frac{\theta}{2}e_{12}}$$

```
[31]: R12 = lambda theta: math.e**(-theta/2*e12)
      R12(math.pi/2)
[31]: 0.71 - (0.71^e12)
```

And use it like this

```
[32]: a = e1+e2+e3
      R = R12(math.pi/2)
      R*a~R
[32]: -(1.0^e1) + (1.0^e2) + (1.0^e3)
```

You might as well make the angle argument a bivector, so that you can control the plane of rotation as well as the angle

$$R_B = e^{-\frac{B}{2}}$$

```
[33]: R_B = lambda B: math.e**(-B/2)
```

Then you could do

```
[34]: R12 = R_B(math.pi/4*e12)
      R23 = R_B(math.pi/5*e23)
```

or

```
[35]: R_B(math.pi/6*(e23+e12)) # rotor enacting a pi/6-rotation in the e23+e12-plane
[35]: 0.93 - (0.26^e12) - (0.26^e23)
```

Maybe you want to define a function which returns a *function* that enacts a specified rotation,

$$f(B) \rightarrow \underline{R_B}(a) = R_B a \tilde{R}_B$$

This just saves you having to write out the sandwich product, which is nice if you are cascading a bunch of rotors, like so

$$\underline{R}_C(\underline{R}_B(\underline{R}_A(a)))$$

```
[36]: def R_factory(B):
      def apply_rotation(a):
          R = math.e**(-B/2)
          return R*a*~R
      return apply_rotation

      R = R_factory(math.pi/6*(e23+e12)) # this returns a function
      R(a) # which acts on a vector

[36]: (0.52^e1) + (0.74^e2) + (1.48^e3)
```

Then you can do things like

```
[37]: R12 = R_factory(math.pi/3*e12)
      R23 = R_factory(math.pi/3*e23)
      R13 = R_factory(math.pi/3*e13)

      R12(R23(R13(a)))

[37]: (0.41^e1) - (0.66^e2) + (1.55^e3)
```

To make cascading a sequence of rotations as concise as possible, we could define a function which takes a list of bivectors  $A, B, C, \dots$ , and enacts the sequence of rotations which they represent on a some vector  $x$ .

$$f(A, B, C, x) = \underline{R}_A(\underline{R}_B(\underline{R}_C(x)))$$

```
[38]: from functools import reduce

      # a sequence of rotations
      def R_seq(*args):
          *Bs, x = args
          R_lst = [math.e**(-B/2) for B in Bs] # create list of Rotors from list of
          ↪Bivectors
          R = reduce(cf.gp, R_lst) # apply the geometric product to list of Rotors
          return R*x*~R

      # rotation sequence by pi/2-in-e12 THEN pi/2-in-e23
      R_seq(math.pi/2*e23, math.pi/2*e12, e1)

[38]: (1.0^e3)
```

## 7.4 Changing Basis Names

If you want to use different names for your basis as opposed to e's with numbers, supply the `Cl()` with a list of names. For example for a two dimensional GA,

```
[39]: layout, blades = cf.Cl(2, names=['', 'x', 'y', 'i'])
```

```
blades
```

```
[39]: {'': 1, 'x': (1^x), 'y': (1^y), 'i': (1^i)}
```

```
[40]: locals().update(blades)
```

```
[41]: 1*x + 2*y
```

```
[41]: (1^x) + (2^y)
```

```
[42]: 1 + 4*i
```

```
[42]: 1 + (4^i)
```

..... doc/tutorials/g3-algebra-of-space.ipynb ends here.

The following section was generated from docs/tutorials/euler-angles.ipynb .....

## ROTATIONS IN SPACE: EULER ANGLES, MATRICES, AND QUATERNIONS

This notebook demonstrates how to use `clifford` to implement rotations in three dimensions using euler angles, rotation matrices and quaternions. All of these forms are derived from the more general rotor form, which is provided by GA. Conversion from the rotor form to a matrix representation is shown, and takes about three lines of code. We start with euler angles.

### 8.1 Euler Angles with Rotors

A common way to parameterize rotations in three dimensions is through [Euler Angles](#).

Any orientation can be achieved by composing three elemental rotations. The elemental rotations can either occur about the axes of the fixed coordinate system (*extrinsic rotations*) or about the axes of a rotating coordinate system, which is initially aligned with the fixed one, and modifies its orientation after each elemental rotation (*intrinsic rotations*). — wikipedia

The animation below shows an intrinsic rotation model as each elemental rotation is applied. Label the left, right, and vertical blue-axes as  $e_1$ ,  $e_2$ , and  $e_3$ , respectively. The series of rotations can be described:

1. rotate about  $e_3$ -axis
2. rotate about the rotated  $e_1$ -axis, call it  $e'_1$
3. rotate about the twice rotated axis of  $e_3$ -axis, call it  $e''_3$

So the elemental rotations are about  $e_3, e'_1, e''_3$ -axes, respectively.

(taken from wikipedia)

Following Sec. 2.7.5 from [DL03], we first rotate an angle  $\phi$  about the  $e_3$ -axis, which is equivalent to rotating in the  $e_{12}$ -plane. This is done with the rotor

$$R_\phi = e^{-\frac{\phi}{2}e_{12}}$$

Next we rotate about the rotated  $e_1$ -axis, which we label  $e'_1$ . To find where this is, we can rotate the axis,

$$e'_1 = R_\phi e_1 \tilde{R}_\phi$$

The plane corresponding to this axis is found by taking the dual of  $e'_1$

$$IR_\phi e_1 \tilde{R}_\phi = R_\phi e_{23} \tilde{R}_\phi$$

Where we have made use of the fact that the pseudo-scalar commutes in G3. Using this result, the second rotation by angle  $\theta$  about the  $e'_1$ -axis is then ,

$$R_\theta = e^{\frac{\theta}{2}R_\phi e_{23} \tilde{R}_\phi}$$

However, noting that

$$e^{R_\phi e_{23}} \tilde{R}_\phi = R_\phi e^{e_{23}} \tilde{R}_\phi$$

Allows us to write the second rotation by angle  $\theta$  about the  $e'_1$ -axis as

$$R_\theta = R_\phi e^{\frac{\theta}{2} e_{23}} \tilde{R}_\phi$$

So, the combination of the first two elemental rotations equals,

$$\begin{aligned} R_\theta R_\phi &= R_\phi e^{\frac{\theta}{2} e_{23}} \tilde{R}_\phi R_\phi \\ &= e^{-\frac{\phi}{2} e_{12}} e^{-\frac{\theta}{2} e_{23}} \end{aligned}$$

This pattern can be extended to the third elemental rotation of angle  $\psi$  in the twice-rotated  $e_1$ -axis, creating the total rotor

$$R = e^{-\frac{\phi}{2} e_{12}} e^{-\frac{\theta}{2} e_{23}} e^{-\frac{\psi}{2} e_{12}}$$

## 8.2 Implementation of Euler Angles

First, we initialize the algebra and assign the variables

```
[1]: from numpy import e, pi
      from clifford import Cl

      layout, blades = Cl(3)    # create a 3-dimensional clifford algebra

      locals().update(blades)  # lazy way to put entire basis in the namespace
```

Next we define a function to produce a rotor given euler angles

```
[2]: def R_euler(phi, theta, psi):
      Rphi = e**(-phi/2.*e12)
      Rtheta = e**(-theta/2.*e23)
      Rpsi = e**(-psi/2.*e12)

      return Rphi*Rtheta*Rpsi
```

For example, using this to create a rotation similar to that shown in the animation above,

```
[3]: R = R_euler(pi/4, pi/4, pi/4)
      R
[3]: 0.65328 - (0.65328^e12) - (0.38268^e23)
```

## 8.3 Convert to Quaternions

A Rotor in 3D space is a *unit quaternion*, and so we have essentially created a function that converts Euler angles to quaternions. All you need to do is interpret the bivectors as  $i$ ,  $j$ , and  $k$ 's. See [Interfacing Other Mathematical Systems](#), for more on quaternions.



## 8.4 Convert to Rotation Matrix

The matrix representation for a rotation can be defined as the result of rotating an ortho-normal frame. Rotating an ortho-normal frame can be done easily,

```
[4]: A = [e1,e2,e3]          # initial ortho-normal frame
      B = [R*a~R for a in A] # resultant frame after rotation

      B

[4]: [(0.14645^e1) + (0.85355^e2) + (0.5^e3),
      -(0.85355^e1) - (0.14645^e2) + (0.5^e3),
      (0.5^e1) - (0.5^e2) + (0.70711^e3)]
```

The components of this frame **are** the rotation matrix, so we just enter the frame components into a matrix.

```
[5]: from numpy import array

      M = [float(b|a) for b in B for a in A] # you need float() due to bug in clifford
      M = array(M).reshape(3,3)

      M

[5]: array([[ 0.14644661,  0.85355339,  0.5          ],
            [-0.85355339, -0.14644661,  0.5          ],
            [ 0.5          , -0.5          ,  0.70710678]])
```

Thats a rotation matrix.

## 8.5 Convert a Rotation Matrix to a Rotor

In 3 Dimenions, there is a simple formula which can be used to directly transform a rotations matrix into a rotor. For arbitrary dimensions you have to use a different algorithm (see `clifford.tools.orthoMat2Versor()` ([docs](#))). Anyway, in 3 dimensions there is a closed form solution, as described in Sec. 4.3.3 of “Geometric Algebra for Physicists”. Given a rotor  $R$  which transforms an orthonormal frame  $A = a_k$  into  $B = b_k$  as such,

$$b_k = Ra_k\tilde{R}$$

$R$  is given by

$$R = \frac{1 + a_k b_k}{|1 + a_k b_k|}$$

So, if you want to convert from a rotation matrix into a rotor, start by converting the matrix M into a frame  $B$ . (You could do this with loop if you want.)

```
[6]: B = [M[0,0]*e1 + M[1,0]*e2 + M[2,0]*e3,
          M[0,1]*e1 + M[1,1]*e2 + M[2,1]*e3,
          M[0,2]*e1 + M[1,2]*e2 + M[2,2]*e3]

      B

[6]: [(0.14645^e1) - (0.85355^e2) + (0.5^e3),
      (0.85355^e1) - (0.14645^e2) - (0.5^e3),
      (0.5^e1) + (0.5^e2) + (0.70711^e3)]
```

Then implement the formula

```
[7]: A = [e1,e2,e3]
      R = 1+sum([A[k]*B[k] for k in range(3)])
      R = R/abs(R)
```

R

```
[7]: 0.65328 - (0.65328^e12) - (0.38268^e23)
```

blam.

..... doc/tutorials/euler-angles.ipynb ends here.

The following section was generated from docs/tutorials/space-time-algebra.ipynb .....

## SPACE TIME ALGEBRA

### 9.1 Intro

This notebook demonstrates how to use `clifford` to work with Space Time Algebra. The Pauli algebra of space  $\mathbb{P}$ , and Dirac algebra of space-time  $\mathbb{D}$ , are related using the *spacetime split*. The split is implemented by using a `BladeMap` ([docs](#)), which maps a subset of blades in  $\mathbb{D}$  to the blades in  $\mathbb{P}$ . This *split* allows a spacetime bivector  $F$  to be broken up into relative electric and magnetic fields in space. Lorentz transformations are implemented as rotations in  $\mathbb{D}$ , and the effects on the relative fields are computed with the split.

### 9.2 Setup

First we import `clifford`, instantiate the two algebras, and populate the namespace with the blades of each algebra. The elements of  $\mathbb{D}$  are prefixed with  $d$ , while the elements of  $\mathbb{P}$  are prefixed with  $p$ . Although unconventional, it is easier to read and to translate into code.

```
[1]: from clifford import Cl, pretty

pretty(precision=1)

# Dirac Algebra `D`
D, D_blades = Cl(1,3, firstIdx=0, names='d')

# Pauli Algebra `P`
P, P_blades = Cl(3, names='p')

# put elements of each in namespace
locals().update(D_blades)
locals().update(P_blades)
```

### 9.3 The Space Time Split

To two algebras can be related by the spacetime-split. First, we create a `BladeMap` which relates the bivectors in  $\mathbb{D}$  to the vectors/bivectors in  $\mathbb{P}$ . The scalars and pseudo-scalars in each algebra are equated.

```
[2]: from clifford import BladeMap

bm = BladeMap([(d01,p1),
```

(continues on next page)

(continued from previous page)

```
(d02,p2),
(d03,p3),
(d12,p12),
(d23,p23),
(d13,p13),
(d0123, p123)])
```

## 9.4 Splitting a space-time vector (an event)

A vector in  $\mathbb{D}$ , represents a unique place in space and time, i.e. an event. To illustrate the split, create a random event  $X$ .

```
[3]: X = D.randomV()*10
X
[3]: -(18.4^d0) - (14.2^d1) - (14.3^d2) - (16.5^d3)
```

This can be *split* into time and space components by multiplying with the time-vector  $d_0$ ,

```
[4]: X*d0
[4]: -18.4 + (14.2^d01) + (14.3^d02) + (16.5^d03)
```

and applying the BladeMap, which results in a scalar+vector in  $\mathbb{P}$

```
[5]: bm(X*d0)
[5]: -18.4 + (14.2^p1) + (14.3^p2) + (16.5^p3)
```

The space and time components can be separated by grade projection,

```
[6]: x = bm(X*d0)
x(0) # the time component
[6]: -18.4

[7]: x(1) # the space component
[7]: (14.2^p1) + (14.3^p2) + (16.5^p3)
```

We therefor define a `split()` function, which has a simple condition allowing it to act on a vector or a multivector in  $\mathbb{D}$ . Splitting a spacetime bivector will be treated in the next section.

```
[8]: def split(X):
      return bm(X.odd*d0+X.even)

[9]: split(X)
[9]: -18.4 + (14.2^p1) + (14.3^p2) + (16.5^p3)
```

The split can be inverted by applying the BladeMap again, and multiplying by  $d_0$

```
[10]: x = split(X)
bm(x)*d0
```

```
[10]: -(18.4^d0) - (14.2^d1) - (14.3^d2) - (16.5^d3)
```

## 9.5 Splitting a Bivector

Given a random bivector  $F$  in  $\mathbb{D}$ ,

```
[11]: F = D.randomMV() (2)
F
```

```
[11]: -(1.2^d01) + (0.2^d02) - (1.3^d03) - (1.6^d12) - (0.0^d13) - (0.2^d23)
```

$F$  splits into a vector/bivector in  $\mathbb{P}$

```
[12]: split(F)
```

```
[12]: -(1.2^p1) + (0.2^p2) - (1.3^p3) - (1.6^p12) - (0.0^p13) - (0.2^p23)
```

If  $F$  is interpreted as the electromagnetic bivector, the Electric and Magnetic fields can be separated by grade

```
[13]: E = split(F) (1)
      iB = split(F) (2)

      E
```

```
[13]: -(1.2^p1) + (0.2^p2) - (1.3^p3)
```

```
[14]: iB
```

```
[14]: -(1.6^p12) - (0.0^p13) - (0.2^p23)
```

## 9.6 Lorentz Transformations

Lorentz Transformations are rotations in  $\mathbb{D}$ , which are implemented with Rotors. A rotor in  $G_4$  will, in general, have scalar, bivector, and quadvector components.

```
[15]: R = D.randomRotor()
R
```

```
[15]: -2.1 - (0.2^d01) - (0.1^d02) - (2.1^d03) + (1.0^d12) + (1.1^d13) - (0.0^d23) + (0.9^
      ↪d0123)
```

In this way, the effect of a lorentz transformation on the electric and magnetic fields can be computed by rotating the bivector with  $F \rightarrow RF\tilde{R}$

```
[16]: F_ = R*F*~R
F_
```

```
[16]: -(15.4^d01) + (6.7^d02) + (8.9^d03) - (2.0^d12) - (17.8^d13) + (6.4^d23)
```

Then splitting into  $E$  and  $B$  fields

```
[17]: E_ = split(F_) (1)
E_
```

```
[17]: -(15.4^p1) + (6.7^p2) + (8.9^p3)
```

```
[18]: iB_ = split(F_)(2)
      iB_
```

```
[18]: -(2.0^p12) - (17.8^p13) + (6.4^p23)
```

## 9.7 Lorentz Invariants

Since lorentz rotations in  $\mathbb{D}$ , the magnitude of elements of  $\mathbb{D}$  are invariants of the lorentz transformation. For example, the magnitude of electromagnetic bivector  $F$  is invariant, and it can be related to  $E$  and  $B$  fields in  $\mathbb{P}$  through the split,

```
[19]: i = p123
      E = split(F)(1)
      B = -i*split(F)(2)
```

```
[20]: F**2
```

```
[20]: 0.5 + (4.6^d0123)
```

```
[21]: split(F**2) == E**2 - B**2 + (2*E|B)*i
```

```
[21]: True
```

..... doc/tutorials/space-time-algebra.ipynb ends here.

The following section was generated from docs/tutorials/InterfacingOtherMathSystems.ipynb .....

## INTERFACING OTHER MATHEMATICAL SYSTEMS

Geometric Algebra is known as a *universal algebra* because it subsumes several other mathematical systems. Two algebras commonly used by engineers and scientists are complex numbers and quaternions. These algebras can be subsumed as the even sub-algebras of 2 and 3 dimensional geometric algebras, respectively. This notebook demonstrates how `clifford` can be used to incorporate data created with these systems into geometric algebra.

### 10.1 Complex Numbers

Given a two dimensional GA with the orthonormal basis,

$$e_i \cdot e_j = \delta_{ij}$$

The geometric algebra consists of scalars, two vectors, and a bivector,

$$\left\{ \underbrace{\alpha}_{\text{scalar}}, \underbrace{e_1, e_2}_{\text{vector}}, \underbrace{e_{12}}_{\text{bivector}} \right\}$$

A complex number can be directly associated with a 2D spinor in the  $e_{12}$ -plane,

$$\underbrace{z = \alpha + \beta i}_{\text{complex number}} \implies \underbrace{Z = \alpha + \beta e_{12}}_{\text{2D spinor}}$$

The even subalgebra of a two dimensional geometric algebra is isomorphic to the complex numbers. We can setup translating functions which converts a 2D spinor into a complex number and vice-versa. In two dimensions the spinor can be also be mapped into vectors if desired.

Below is an illustration of the three different planes, the later two being contained within the geometric algebra of two dimensions,  $G_2$ . Both spinors and vectors in  $G_2$  can be modeled as points on a plane, but they have distinct algebraic properties.

```
[1]: import clifford as cf
layout, blades = cf.Cl(2) # instantiate a 2D- GA
locals().update(blades)  # put all blades into local namespace

def c2s(z):
    '''convert a complex number to a spinor'''
    return z.real + z.imag*e12

def s2c(S):
    '''convert a spinor to a complex number'''
    S0 = float(S(0))
    S2 = float(-S(e12))
    return S0 + S2*1j
```

Convert a complex number to a spinor

```
[2]: c2s(1+2j)
[2]: 1.0 + (2.0^e12)
```

Convert a spinor to a complex number

```
[3]: s2c(1+2*e12)
[3]: (1+2j)
```

Make sure we get what we started with when we make a round trip

```
[4]: s2c(c2s(1+2j)) == 1+2j
[4]: True
```

The spinor is then mapped to a vector by choosing a reference direction. This may be done by left multiplying with  $e_1$ .

$$Z \implies e_1 Z = e_1 \alpha + \beta e_1 e_{12} = \underbrace{\alpha e_1 + \beta e_2}_{\text{vector}}$$

```
[5]: s = 1+2*e12
     e1*s
[5]: (1^e1) + (2^e2)
```

Geometrically, this is interpreted as having the spinor rotate a specific vector, in this case  $e_1$ . Building off of the previously defined functions

```
[6]: def c2v(c):
     '''convert a complex number to a vector'''
     return e1*c2s(c)

     def v2c(v):
     '''convert a vector to a complex number'''
     return s2c(e1*v)
```

```
[7]: c2v(1+2j)
[7]: (1.0^e1) + (2.0^e2)
```

```
[8]: v2c(1*e1+2*e2)
[8]: (1+2j)
```

Depending on your applications, you may wish to have the bivector be an argument to the `c2s` and `s2c` functions. This allows you to map input data given in the form of complex number onto the planes of your choice. For example, in three dimensional space there are three bivector-planes;  $e_{12}$ ,  $e_{23}$  and  $e_{13}$ , so there are many bivectors which could be interpreted as the unit imaginary.

Complex numbers mapped in this way can be used to enact rotations within the specified planes.

```
[9]: import clifford as cf
     layout, blades = cf.Cl(3)
     locals().update(blades)
```

(continues on next page)



(continued from previous page)

```
def c2s(z,B):
    '''convert a complex number to a spinor'''
    return z.real + z.imag*B

def s2c(S,B):
    '''convert a spinor to a complex number'''
    S0 = float(S(0))
    S2 = float(-S|B)
    return S0 + S2*1j
```

```
[10]: c2s(1+2j, e23)
```

```
[10]: 1.0 + (2.0^e23)
```

```
[11]: c2s(3+4j, e13)
```

```
[11]: 3.0 + (4.0^e13)
```

This brings us to the subject of quaternions, which are used to handle rotations in three dimensions much like complex numbers do in two dimensions. With geometric algebra, they are just spinors acting in a different geometry.

## 10.2 Quaternions

### Note:

There is support for quaternions in numpy through the package [quaternion](#).

For some reason people think quaternions ([wiki page](#)) are mystical or something. They are just spinors in a three dimensional geometric algebra.

In either case, we can pass the names parameters to `Cl()` to explicitly label the bivectors  $i$ ,  $j$ , and  $k$ .

```
[12]: import clifford as cf

# the vector/bivector order is reversed because Hamilton defined quaternions using a
# left-handed frame. wtf.
names = ['', 'z', 'y', 'x', 'k', 'j', 'i', 'I']

layout, blades = cf.Cl(3, names=names)
locals().update(blades)
```

This leads to the commutations relations familiar to quaternion users

```
[13]: for m in [i, j, k]:
        for n in [i, j, k]:
            print ('{}*{}={}'.format(m, n, m*n))

(1^i)*(1^i)=-1
(1^i)*(1^j)=(1^k)
(1^i)*(1^k)=-1
(1^j)*(1^i)=-1
(1^j)*(1^j)=-1
(1^j)*(1^k)=(1^i)
```

(continues on next page)

(continued from previous page)

```
(1^k) * (1^i) = (1^j)
(1^k) * (1^j) = -(1^i)
(1^k) * (1^k) = -1
```

Quaternion data could be stored in a variety of ways. Assuming you have the scalar components for the quaternion, all you will need to do is setup a map each component to the correct bivector.

```
[14]: def q2S(*args):
      '''convert tuple of quaternion coefficients to a spinor'''
      q = args
      return q[0] + q[1]*i + q[2]*j + q[3]*k
```

Then all the quaternion computations can be done using GA

```
[15]: q1 = q2S(1, 2, 3, 4)
      q1
```

```
[15]: 1 + (4^k) + (3^j) + (2^i)
```

This prints  $i$ ,  $j$  and  $k$  in reverse order but whatever,

```
[16]: # 'scalar' part
      q1(0)
```

```
[16]: 1
```

```
[17]: # 'vector' part (more like bivector part!)
      q1(2)
```

```
[17]: (4^k) + (3^j) + (2^i)
```

*quaternion conjugation* is implemented with reversion

```
[18]: ~q1
```

```
[18]: 1 - (4^k) - (3^j) - (2^i)
```

The norm

```
[19]: abs(q1)
```

```
[19]: 5.477225575051661
```

Taking the `dual()` of the “vector” part actually returns a vector,

```
[20]: q1(2).dual()
```

```
[20]: (2^z) - (3^y) + (4^x)
```

```
[21]: q1 = q2S(1, 2, 3, 4)
      q2 = q2S(5, 6, 7, 8)
```

```
# quaternion product
q1*q2
```

```
[21]: -60 + (24^k) + (30^j) + (12^i)
```

If you want to keep using a left-handed frame and names like  $i$ ,  $j$  and  $k$  to label the planes in 3D space, ok. If you think it makes more sense to use the consistent and transparent approach provided by GA, we think you have good taste. If we make this switch, the basis and `q2S()` functions will be changed to

```
[22]: import clifford as cf
      layout, blades = cf.Cl(3)
      locals().update(blades)
```

```
blades
```

```
[22]: {'': 1,
      'e1': (1^e1),
      'e2': (1^e2),
      'e3': (1^e3),
      'e12': (1^e12),
      'e13': (1^e13),
      'e23': (1^e23),
      'e123': (1^e123)}
```

```
[23]: def q2S(*args):
      '''
      convert tuple of quaternion coefficients to a spinor'''
      q = args
      return q[0] + q[1]*e13 + q[2]*e23 + q[3]*e12

      q1 = q2S(1,2,3,4)
      q1
```

```
[23]: 1 + (4^e12) + (2^e13) + (3^e23)
```

..... doc/tutorials/InterfacingOtherMathSystems.ipynb ends here.

The following section was generated from docs/tutorials/PerformanceCliffordTutorial.ipynb .....



## WRITING HIGH(ISH) PERFORMANCE CODE WITH CLIFFORD AND NUMBA VIA NUMPY

This document describes how to take algorithms developed in the clifford package with notation that is close to the maths and convert it into numerically efficient and fast running code. To do this we will expose the underlying representation of multivector as a numpy array of canonical basis vector coefficients and operate directly on these arrays in a manner that is conducive to JIT compilation with numba.

### 11.1 First import the Clifford library as well as numpy and numba

```
[1]: import clifford as cf
import numpy as np
import numba
```

### 11.2 Choose a specific space

For this document we will use 3d euclidean space embedded in the conformal framework giving a  $Cl(4,1)$  algebra. We will also rename some of the variables to match the notation that used by Lasenby et al. in “A Covariant Approach to Geometry using Geometric Algebra”

```
[2]: from clifford import g3c
# Get the layout in our local namespace etc etc
layout = g3c.layout
locals().update(g3c.blades)

ep, en, up, down, homo, E0, ninf, no = (g3c.stuff["ep"], g3c.stuff["en"],
                                         g3c.stuff["up"], g3c.stuff["down"], g3c.stuff[
↳ "homo"],
                                         g3c.stuff["E0"], g3c.stuff["einf"], -g3c.
↳ stuff["eo"])
# Define a few useful terms
E = ninf^(no)
I5 = e12345
I3 = e123
```

## 11.3 Performance of mathematically idiomatic Clifford algorithms

By default the Clifford library sacrifices performance for syntactic convenience.

Consider a function that applies a rotor to a multivector:

```
[3]: def apply_rotor(R, mv):
      return R*mv*~R
```

We will define a rotor that takes one line to another:

```
[4]: line_one = (up(0)^up(e1)^ninf).normal()
      line_two = (up(0)^up(e2)^ninf).normal()
      R = 1 + line_two*line_one
```

Check that this works

```
[5]: print(line_two)
      print(apply_rotor(R, line_one).normal())

(1.0^e245)
(1.0^e245)
```

We would like to improve the speed of our algorithm, first we will profile it and see where it spends its time

```
[6]: %%prun -s cumtime
      #for i in range(1000000):
      #    apply_rotor(R, line_one)
```

An example profile output from running this notebook on the author's laptop is as follows:

ncalls	tottime	percall	cumtime	percall	filename:lineno(function)
1	0.000	0.000	66.290	66.290	{built-in method builtins.exec}
1	0.757	0.757	66.290	66.290	<string>:2(<module>)
1000000	3.818	0.000	65.534	0.000	<ipython-input-13-70a01003bf51>:1(apply_rotor)
2000000	9.269	0.000	55.641	0.000	__init__.py:751(__mul__)
2000000	3.167	0.000	29.900	0.000	__init__.py:717(_checkOther)
2000000	1.371	0.000	19.906	0.000	__init__.py:420(__ne__)
2000000	6.000	0.000	18.535	0.000	numeric.py:2565(array_equal)
2000000	10.505	0.000	10.505	0.000	__init__.py:260(mv_mult)

We can see that the function spends almost all of its time in `__mul__` and within `__mul__` it spends most of its time in `_checkOther`. In fact it only spends a small fraction of its time in `mv_mult` which does the numerical multivector multiplication. To write more performant code we need to strip away the high level abstractions and deal with the underlying representations of the blade component data.

## 11.4 Canonical blade coefficient representation in Clifford

In Clifford a multivector is internally represented as a numpy array of the coefficients of the canonical basis vectors, they are arranged in order of grade. So for our 4,1 algebra the first element is the scalar part, the next 5 are the vector coefficients, the next 10 are the bivectors, the next 10 the triectors, the next 5 the quadvectors and the final value is the pseudoscalar coefficient.

```
[7]: (5.0*e1 - e2 + e12 + e135 + np.pi*e1234).value
[7]: array([ 0.          ,  5.          , -1.          ,  0.          ,  0.          ,
          0.          ,  1.          ,  0.          ,  0.          ,  0.          ,
          0.          ,  0.          ,  0.          ,  0.          ,  0.          ,
          1.          ,  0.          ,  0.          ,  0.          ,  0.          ,
          0.          ,  3.14159265,  0.          ,  0.          ,  0.          ,
          0.          ,  0.          ])
```

## 11.5 Exploiting blade representation to write a fast function

We can rewrite our rotor application function using the functions that the layout exposes for operations on the numpy arrays themselves.

```
[8]: def apply_rotor_faster(R,mv):
      return layout.MultiVector(layout.gmt_func(R.value,layout.gmt_func(mv.value,layout.
      ↪adjoint_func(R.value))) )

[9]: %%%prun -s cumtime
     #for i in range(1000000):
     #    apply_rotor_faster(R,line_one)
```

This gives a much faster function

ncalls	tottime	percall	cumtime	percall	filename:lineno(function)
1	0.000	0.000	19.567	19.567	{built-in method builtins.exec}
1	0.631	0.631	19.567	19.567	<string>:2 (<module>)
1000000	7.373	0.000	18.936	0.000	<ipython-input-35-6a5344d83bdb>:1 (apply_ ↪ rotor_faster)
2000000	9.125	0.000	9.125	0.000	__init__.py:260 (mv_mult)
1000000	1.021	0.000	1.619	0.000	__init__.py:677 (__init__)
1000000	0.819	0.000	0.819	0.000	__init__.py:244 (adjoint_func)

We have successfully skipped past the higher level checks on the multivectors while maintaining exactly the same function signature. It is important to check that we still have the correct answer:

```
[10]: print(line_two)
      print(apply_rotor_faster(R,line_one).normal())

(1.0^e245)
(1.0^e245)
```

The performance improvements gained by rewriting our function are significant but it comes at the cost of readability.

By loading the layouts `gmt_func` and `adjoint_func` into the global namespace before the function is defined and separating the value operations from the multivector wrapper we can make our code more concise.

```
[11]: gmt_func = layout.gmt_func
adjoint_func = layout.adjoint_func

def apply_rotor_val(R_val, mv_val):
    return gmt_func(R_val, gmt_func(mv_val, adjoint_func(R_val)))

def apply_rotor_wrapped(R, mv):
    return cf.MultiVector(layout, apply_rotor_val(R.value, mv.value))

[12]: %%prun -s cumtime
# for i in range(1000000):
#     apply_rotor_wrapped(R, line_one)
```

The time cost is essentially the same, there is probably some minor overhead from the function call itself

ncalls	totttime	percall	cumtime	percall	filename:lineno(function)
1	0.000	0.000	19.621	19.621	{built-in method builtins.exec}
1	0.557	0.557	19.621	19.621	<string>:2(<module>)
1000000	1.421	0.000	19.064	0.000	<ipython-input-38-a1e0b5c53cdc>:7(apply_rotor_wrapped)
1000000	6.079	0.000	16.033	0.000	<ipython-input-38-a1e0b5c53cdc>:4(apply_rotor_val)
2000000	9.154	0.000	9.154	0.000	__init__.py:260(mv_mult)
1000000	1.017	0.000	1.610	0.000	__init__.py:677(__init__)
1000000	0.800	0.000	0.800	0.000	__init__.py:244(adjoint_func)

```
[13]: print(line_two)
print(apply_rotor_wrapped(R, line_one).normal())

(1.0^e245)
(1.0^e245)
```

The additional advantage of splitting the function like this is that the numba JIT compiler can reason about the memory layout of numpy arrays in no python mode as long as no pure python objects are operated upon within the function. This means we can JIT our function that operates on the value directly.

```
[14]: @numba.njit
def apply_rotor_val_numba(R_val, mv_val):
    return gmt_func(R_val, gmt_func(mv_val, adjoint_func(R_val)))

def apply_rotor_wrapped_numba(R, mv):
    return cf.MultiVector(layout, apply_rotor_val_numba(R.value, mv.value))

[15]: %%prun -s cumtime
# for i in range(1000000):
#     apply_rotor_wrapped_numba(R, line_one)
```

This gives a small improvement in performance but more importantly it allows us to write larger functions that also use the jitted `apply_rotor_val_numba` and are themselves jitted.

ncalls	totttime	percall	cumtime	percall	filename:lineno(function)
1	0.000	0.000	16.033	16.033	{built-in method builtins.exec}
1	0.605	0.605	16.033	16.033	<string>:2(<module>)
1000000	2.585	0.000	15.428	0.000	<ipython-input-42-1142126d93ca>:5(apply_rotor_wrapped_numba)
1000000	8.606	0.000	8.606	0.000	<ipython-input-42-1142126d93ca>:1(apply_rotor_val_numba)

(continues on next page)



(continued from previous page)

1	0.000	0.000	2.716	2.716	dispatcher.py:294 (_compile_for_args)
7/1	0.000	0.000	2.716	2.716	dispatcher.py:554 (compile)

## 11.6 Composing larger functions

By chaining together functions that operate on the value arrays of multivectors it is easy to construct fast and readable code

```
[16]: I5_val = I5.value
      omt_func = layout.omt_func

      def dual_mv(mv):
          return -I5*mv

      def meet_unwrapped(mv_a, mv_b):
          return -dual_mv(dual_mv(mv_a) ^ dual_mv(mv_b))

      @numba.njit
      def dual_val(mv_val):
          return -gmt_func(I5_val, mv_val)

      @numba.njit
      def meet_val(mv_a_val, mv_b_val):
          return -dual_val(omt_func(dual_val(mv_a_val), dual_val(mv_b_val)))

      def meet_wrapped(mv_a, mv_b):
          return cf.layout.MultiVector(meet_val(mv_a.value, mv_b.value))

      sphere = (up(0)^up(e1)^up(e2)^up(e3)).normal()
      print(sphere.meet(line_one).normal().normal())
      print(meet_unwrapped(sphere, line_one).normal())
      print(meet_wrapped(line_one, sphere).normal())

      (1.0^e14) - (1.0^e15) - (1.0^e45)
      (1.0^e14) - (1.0^e15) - (1.0^e45)

      -----
      AttributeError                                Traceback (most recent call last)
      <ipython-input-1-059d06919085> in <module>
          22 print(sphere.meet(line_one).normal().normal())
          23 print(meet_unwrapped(sphere, line_one).normal())
      --> 24 print(meet_wrapped(line_one, sphere).normal())

      <ipython-input-1-059d06919085> in meet_wrapped(mv_a, mv_b)
          17
          18 def meet_wrapped(mv_a, mv_b):
      --> 19     return cf.layout.MultiVector(meet_val(mv_a.value, mv_b.value))
          20
          21 sphere = (up(0)^up(e1)^up(e2)^up(e3)).normal()

      AttributeError: module 'clifford' has no attribute 'layout'
```

```
[17]: %%prun -s cumtime
      #for i in range(100000):
      #    meet_unwrapped(sphere, line_one)
```

```
ncalls tottime percall cumtime percall filename:lineno(function) 1 0.
000 0.000 13.216 13.216 {built-in method builtins.exec} 1 0.085 0.
085 13.216 13.216 <string>:2(<module>) 100000 0.418 0.000 13.131 0.000
<ipython-input-98-f91457c8741a>:7(meet_unwrapped) 300000 0.681 0.000 9.893
0.000 <ipython-input-98-f91457c8741a>:4(dual_mv) 300000 1.383 0.000 8.127
0.000 __init__.py:751(__mul__) 400000 0.626 0.000 5.762 0.000 __init__.py:
717(_checkOther) 400000 0.270 0.000 3.815 0.000 __init__.py:420(__ne__) 400000
1.106 0.000 3.544 0.000 numeric.py:2565(array_equal) 100000 0.460 0.000 2.439
0.000 __init__.py:783(__xor__) 800000 0.662 0.000 2.053 0.000 __init__.py:
740(_newMV) 400000 1.815 0.000 1.815 0.000 __init__.py:260(mv_mult)
```

```
[18]: %%prun -s cumtime
# for i in range(100000):
#     meet_wrapped(sphere, line_one)
```

```
ncalls tottime percall cumtime percall filename:lineno(function) 1
0.000 0.000 1.951 1.951 {built-in method builtins.exec} 1 0.063 0.
063 1.951 1.951 <string>:2(<module>) 100000 0.274 0.000 1.888 0.000
<ipython-input-98-f91457c8741a>:18(meet_wrapped) 100000 1.448 0.000 1.448
0.000 <ipython-input-98-f91457c8741a>:14(meet_val) 100000 0.096 0.000 0.166
0.000 __init__.py:677(__init__)
```

## 11.7 Algorithms exploiting known sparseness of MultiVector value array

The standard multiplication generator function for two general multivectors is as follows:

```
[19]: def get_mult_function(mult_table, n_dims):
    """
    Returns a function that implements the mult_table on two input multivectors
    """
    non_zero_indices = mult_table.nonzero()
    k_list = non_zero_indices[0]
    l_list = non_zero_indices[1]
    m_list = non_zero_indices[2]
    mult_table_vals = np.array([mult_table[k, l, m] for k, l, m in np.transpose(non_zero_
    ↪indices)], dtype=int)

    @numba.njit
    def mv_mult(value, other_value):
        output = np.zeros(n_dims)
        for ind, k in enumerate(k_list):
            l = l_list[ind]
            m = m_list[ind]
            output[l] += value[k] * mult_table_vals[ind] * other_value[m]
        return output
    return mv_mult
```

There are however instances in which we might be able to use the known sparseness of the input data value representation to speed up the operations. For example, in  $Cl(4,1)$  rotors only contain even grade blades and we can therefore remove all the operations accessing odd grade objects.

```
[20]: def get_grade_from_index(index_in):
    if index_in == 0:
```

(continues on next page)

(continued from previous page)

```

        return 0
    elif index_in < 6:
        return 1
    elif index_in < 16:
        return 2
    elif index_in < 26:
        return 3
    elif index_in < 31:
        return 4
    elif index_in == 31:
        return 5
    else:
        raise ValueError('Index is out of multivector bounds')

def get_sparse_mult_function(mult_table, n_dims, grades_a, grades_b):
    """
    Returns a function that implements the mult_table on two input multivectors
    """
    non_zero_indices = mult_table.nonzero()
    k_list = non_zero_indices[0]
    l_list = non_zero_indices[1]
    m_list = non_zero_indices[2]
    mult_table_vals = np.array([mult_table[k,l,m] for k,l,m in np.transpose(non_zero_
↪indices)], dtype=int)

    # Now filter out the sparseness
    filter_mask = np.zeros(len(k_list), dtype=bool)
    for i in range(len(filter_mask)):
        if get_grade_from_index(k_list[i]) in grades_a:
            if get_grade_from_index(m_list[i]) in grades_b:
                filter_mask[i] = 1

    k_list = k_list[filter_mask]
    l_list = l_list[filter_mask]
    m_list = m_list[filter_mask]
    mult_table_vals = mult_table_vals[filter_mask]

    @numba.njit
    def mv_mult(value, other_value):
        output = np.zeros(n_dims)
        for ind, k in enumerate(k_list):
            l = l_list[ind]
            m = m_list[ind]
            output[l] += value[k] * mult_table_vals[ind] * other_value[m]
        return output
    return mv_mult

```

```

[21]: left_rotor_mult = get_sparse_mult_function(layout.gmt, layout.gaDims, [0, 2, 4], [0, 1, 2, 3,
↪4, 5])
right_rotor_mult = get_sparse_mult_function(layout.gmt, layout.gaDims, [0, 1, 2, 3, 4, 5], [0,
↪2, 4])

@numba.njit
def sparse_apply_rotor_val(R_val, mv_val):
    return left_rotor_mult(R_val, right_rotor_mult(mv_val, adjoint_func(R_val)))

```

(continues on next page)

(continued from previous page)

```
def sparse_apply_rotor(R,mv):
    return cf.MultiVector(layout,sparse_apply_rotor_val(R.value,mv.value))
```

```
[22]: %%prun -s cumtime
      #for i in range(1000000):
      #    sparse_apply_rotor(R,line_one)
```

```
ncalls  tottime  percall  cumtime  percall filename:lineno(function)
      1      0.000      0.000      9.490      9.490 {built-in method builtins.exec}
      1      0.624      0.624      9.489      9.489 <string>:2(<module>)
1000000      2.684      0.000      8.865      0.000 <ipython-input-146-f75aae3ce595>:
→8(sparse_apply_rotor)
1000000      4.651      0.000      4.651      0.000 <ipython-input-146-f75aae3ce595>:
→4(sparse_apply_rotor_val)
1000000      0.934      0.000      1.530      0.000 __init__.py:677(__init__)
1000000      0.596      0.000      0.596      0.000 {built-in method numpy.core.multiarray.
→array}
```

```
[23]: print(line_two)
      print(sparse_apply_rotor(R,line_one).normal())

(1.0^e245)
(1.0^e245)
```

We can do the same with the meet operation that we defined earlier if we know what grade objects we are meeting

```
[24]: left_pseudo_mult = get_sparse_mult_function(layout.gmt,layout.gaDims,[5],[0,1,2,3,4,
→5])
      sparse_omt_2_1 = get_sparse_mult_function(layout.omt,layout.gaDims,[2],[1])

@numba.njit
def dual_sparse_val(mv_val):
    return -left_pseudo_mult(I5_val,mv_val)

@numba.njit
def meet_sparse_3_4_val(mv_a_val,mv_b_val):
    return -dual_sparse_val(sparse_omt_2_1(dual_sparse_val(mv_a_val), dual_sparse_
→val(mv_b_val)))

def meet_sparse_3_4(mv_a,mv_b):
    return cf.layout.MultiVector(meet_sparse_3_4_val(mv_a.value, mv_b.value))

print(sphere.meet(line_one).normal().normal())
print(meet_sparse_3_4(line_one,sphere).normal())

(1.0^e14) - (1.0^e15) - (1.0^e45)
```

```
-----
AttributeError                                Traceback (most recent call last)
<ipython-input-1-b25f7cce90d8> in <module>
     14
     15 print(sphere.meet(line_one).normal().normal())
--> 16 print(meet_sparse_3_4(line_one,sphere).normal())

<ipython-input-1-b25f7cce90d8> in meet_sparse_3_4(mv_a, mv_b)
     11
     12 def meet_sparse_3_4(mv_a,mv_b):
```

(continues on next page)

(continued from previous page)

```

--> 13     return cf.layout.MultiVector(meet_sparse_3_4_val(mv_a.value, mv_b.value))
14
15 print(sphere.meet(line_one).normal().normal())

```

```
AttributeError: module 'clifford' has no attribute 'layout'
```

```

[25]: %%prun -s cumtime
      #for i in range(100000):
      #    meet_sparse_3_4(line_one, sphere)

```

ncalls	totttime	percall	cumtime	percall	filename:lineno(function)
1	0.000	0.000	0.725	0.725	{built-in method builtins.exec}
1	0.058	0.058	0.725	0.725	<string>:2(<module>)
100000	0.252	0.000	0.667	0.000	<ipython-input-156-f346d0563682>:12 (meet_ ↪ sparse_3_4)
100000	0.267	0.000	0.267	0.000	<ipython-input-156-f346d0563682>:8 (meet_ ↪ sparse_3_4_val)
100000	0.088	0.000	0.148	0.000	__init__.py:677(__init__)

## 11.8 Future work on performance

Investigate efficient operations on containers of large numbers of multivectors.

Possibly investigate <http://numba.pydata.org/numba-doc/0.13/CUDAjit.html> for larger algebras/other areas in which GPU memory latency will not be such a large factor, ie, lots of bulk parallel numerical operations.

```

[ ]: ..... doc/tutorials/PerformanceCliffordTutorial.ipynb ends here.

```

The following section was generated from docs/tutorials/cga/index.ipynb .....



## CONFORMAL GEOMETRIC ALGEBRA

Conformal Geometric Algebra (CGA) is a projective geometry tool which allows conformal transformations to be implemented with rotations. To do this, the original geometric algebra is extended by two dimensions, one of positive signature  $e_+$  and one of negative signature  $e_-$ . Thus, if we started with  $G_p$ , the conformal algebra is  $G_{p+1,1}$ .

It is convenient to define a *null* basis given by

$$e_o = \frac{1}{2}(e_- - e_+)$$

$$e_\infty = e_- + e_+$$

A vector in the original space  $x$  is *up-projected* into a conformal vector  $X$  by

$$X = x + \frac{1}{2}x^2e_\infty + e_o$$

To map a conformal vector back into a vector from the original space, the vector is first normalized, then rejected from the minkowski plane  $E_0$ ,

$$X = \frac{X}{X \cdot e_\infty}$$

then

$$x = X \wedge E_0 E_0^{-1}$$

To implement this in `clifford` we could create a CGA by instantiating the it directly, like `Cl(3, 1)` for example, and then making the definitions and maps described above relating the various subspaces. Or, we you can use the helper function `conformalize()`.

### 12.1 Using `conformalize()`

The purpose of `conformalize()` is to remove the redundancy associated with creating a conformal geometric algebras. `conformalize()` takes an existing geometric algebra layout and *conformalizes* it by adding two dimensions, as described above. Additionally, this function returns a new layout for the CGA, a dict of blades for the CGA, and dictionary containing the added basis vectors and up/down projection functions.

To demonstrate we will conformalize  $G_2$ , producing a CGA of  $G_{3,1}$ .

```
[1]: from numpy import pi, e
      from clifford import Cl, conformalize

      G2, blades_g2 = Cl(2)

      blades_g2 # inspect the G2 blades
```

```
[1]: {'': 1, 'e1': (1^e1), 'e2': (1^e2), 'e12': (1^e12)}
```

Now, conformalize it

```
[2]: G2c, blades_g2c, stuff = conformalize(G2)
```

```
blades_g2c    #inspect the CGA blades
```

```
[2]: {'': 1,
      'e1': (1^e1),
      'e2': (1^e2),
      'e3': (1^e3),
      'e4': (1^e4),
      'e12': (1^e12),
      'e13': (1^e13),
      'e14': (1^e14),
      'e23': (1^e23),
      'e24': (1^e24),
      'e34': (1^e34),
      'e123': (1^e123),
      'e124': (1^e124),
      'e134': (1^e134),
      'e234': (1^e234),
      'e1234': (1^e1234)}
```

Additionally lets inspect stuff

```
[3]: stuff
```

```
[3]: {'ep': (1^e3),
      'en': (1^e4),
      'eo': -(0.5^e3) + (0.5^e4),
      'einf': (1^e3) + (1^e4),
      'E0': (1.0^e34),
      'up': <bound method ConformalLayout.up of ConformalLayout([1, 1, 1, -1],
↳ids=BasisVectorIds.ordered_integers(4), order=BasisBladeOrder.shortlex(4), names=['
↳', 'e1', 'e2', 'e3', 'e4', 'e12', 'e13', 'e14', 'e23', 'e24', 'e34', 'e123', 'e124',
↳ 'e134', 'e234', 'e1234'])>,
      'down': <bound method ConformalLayout.down of ConformalLayout([1, 1, 1, -1],
↳ids=BasisVectorIds.ordered_integers(4), order=BasisBladeOrder.shortlex(4), names=['
↳', 'e1', 'e2', 'e3', 'e4', 'e12', 'e13', 'e14', 'e23', 'e24', 'e34', 'e123', 'e124',
↳ 'e134', 'e234', 'e1234'])>,
      'homo': <bound method ConformalLayout.homo of ConformalLayout([1, 1, 1, -1],
↳ids=BasisVectorIds.ordered_integers(4), order=BasisBladeOrder.shortlex(4), names=['
↳', 'e1', 'e2', 'e3', 'e4', 'e12', 'e13', 'e14', 'e23', 'e24', 'e34', 'e123', 'e124',
↳ 'e134', 'e234', 'e1234'])>,
      'I_base': (1.0^e12)}
```

It contains the following:

- ep - positive basis vector added
- en - negative basis vector added
- eo - zero vector of null basis ( $=.5*(en-ep)$ )
- einf - infinity vector of null basis ( $=en+ep$ )
- E0 - minkowski bivector ( $=einf^eo$ )
- up () - function to up-project a vector from GA to CGA



- `down()` - function to down-project a vector from CGA to GA
- `homo()` - function to homogenize a CGA vector

We can put the `blades` and the `stuff` into the local namespace,

```
[4]: locals().update(blades_g2c)
     locals().update(stuff)
```

Now we can use the `up()` and `down()` functions to go in and out of CGA

```
[5]: x = e1+e2
     X = up(x)
     X
[5]: (1.0^e1) + (1.0^e2) + (0.5^e3) + (1.5^e4)

[6]: down(X)
[6]: (1.0^e1) + (1.0^e2)
```

## 12.2 Operations

Conformal transformations in  $G_n$  are achieved through versors in the conformal space  $G_{n+1,1}$ . These versors can be categorized by their relation to the added minkowski plane,  $E_0$ . There are three categories,

- versor purely in  $E_0$
- versor partly in  $E_0$
- versor out of  $E_0$

A three dimensional projection for conformal space with the relevant subspaces labeled is shown below.

### 12.2.1 Versors purely in $E_0$

First we generate some vectors in  $G_2$ , which we can operate on

```
[7]: a= 1*e1 + 2*e2
     b= 3*e1 + 4*e2
```

#### Inversions

$$e_+ X e_+$$

Inversion is a reflection in  $e_+$ , this swaps  $e_o$  and  $e_\infty$ , as can be seen from the model above.

```
[8]: assert (down(ep*up(a)*ep) == a.inv())
```

## Involutions

$$E_0 X E_0$$

```
[9]: assert (down(E0*up(a)*E0) == -a)
```

## Dilations

$$D_\alpha = e^{-\frac{\ln \alpha}{2} E_0}$$

$$D_\alpha X \tilde{D}_\alpha$$

```
[10]: from scipy import rand, log

D = lambda alpha: e**((-log(alpha)/2.)*(E0))
alpha = rand()
assert (down( D(alpha)*up(a)*~D(alpha)) == (alpha*a))
```

```
<ipython-input-1-0e2e8e068b72>:4: DeprecationWarning: scipy.rand is deprecated and
↳ will be removed in SciPy 2.0.0, use numpy.random.rand instead
    alpha = rand()
<ipython-input-1-0e2e8e068b72>:3: DeprecationWarning: scipy.log is deprecated and
↳ will be removed in SciPy 2.0.0, use numpy.lib.scimath.log instead
    D = lambda alpha: e**((-log(alpha)/2.)*(E0))
```

### 12.2.2 Versors partly in $E_0$

#### Translations

$$V = e^{\frac{1}{2}e_\infty a} = 1 + e_\infty a$$

```
[11]: T = lambda x: e**(1/2.*(einf*x))
assert (down( T(a)*up(b)*~T(a)) == b+a)
```

#### Transversions

A transversion is an inversion, followed by a translation, followed by a inversion. The versor is

$$V = e_+ T_a e_+$$

which is recognised as the translation bivector reflected in the  $e_+$  vector. From the diagram, it is seen that this is equivalent to the bivector in  $x \wedge e_o$ ,

$$e_+(1 + e_\infty a)e_+$$

$$e_+^2 + e_+ e_\infty a e_+$$

$$2 + 2e_o a$$

the factor of 2 may be dropped, because the conformal vectors are null

```
[12]: V = ep * T(a) * ep
      assert ( V == 1+(eo*a) )

      K = lambda x: 1+(eo*a)

      B= up(b)
      assert ( down(K(a)*B*~K(a)) == 1/(a+1/b) )
```

### 12.2.3 Versors Out of $E_0$

Versors that are out of  $E_0$  are made up of the versors within the original space. These include reflections and rotations, and their conformal representation is identical to their form in  $G^n$ , except the minus sign is dropped for reflections,

#### Reflections

$$-mam^{-1} \rightarrow M\tilde{A}\tilde{M}$$

```
[13]: m = 5*e1 + 6*e2
      n = 7*e1 + 8*e2

      assert (down(m*up(a)*m) == -m*a*m.inv())
```

#### Rotations

$$mnanm = Ra\tilde{R} \rightarrow R\tilde{A}\tilde{R}$$

```
[14]: R = lambda theta: e**((-0.5*theta)*(e12))
      theta = pi/2
      assert (down( R(theta)*up(a)*~R(theta)) == R(theta)*a*~R(theta))
```

### 12.2.4 Combinations of Operations

#### simple example

As a simple example consider the combination operations of translation, scaling, and inversion.

$$b = -2a + e_0 \rightarrow B = (T_{e_0}E_0D_2)A(D_2\tilde{E}_0T_{e_0})$$

```
[15]: A = up(a)
      V = T(e1)*E0*D(2)
      B = V*A*~V
      assert (down(B) == (-2*a)+e1 )
```

```
<ipython-input-1-0e2e8e068b72>:3: DeprecationWarning: scipy.log is deprecated and
↳ will be removed in SciPy 2.0.0, use numpy.lib.scimath.log instead
D = lambda alpha: e**((-log(alpha)/2.)*(E0))
```

## Transversion

A transversion may be built from a inversion, translation, and inversion.

$$c = (a^{-1} + b)^{-1}$$

In conformal GA, this is accomplished by

$$C = VA\tilde{V}$$

$$V = e_+ T_b e_+$$

```
[16]: A = up(a)
V = ep*T(b)*ep
C = V*A*~V
assert (down(C) == 1/(1/a + b))
```

## Rotation about a point

Rotation about a point  $a$  can be achieved by translating the origin to  $a$ , then rotating, then translating back. Just like the transversion can be thought of as translating the involution operator, rotation about a point can also be thought of as translating the Rotor itself. Covariance.

## 12.3 More examples

The following section was generated from docs/tutorials/cga/visualization-tools.ipynb .....

### 12.3.1 Visualization tools

In this example we will look at some external tools that can be used with `clifford` to help visualize geometric objects.

The two tools available are:

- `pyganja` ([github](#))
- `mpl_toolkits.clifford` ([github](#))

Both of these can be installed with `pip install` followed by the package name above.

## G2C

Let's start by creating some objects in 2d Conformal Geometric Algebra to visualize:

```
[1]: from clifford.g2c import *

[2]: point = up(2*e1+e2)
line = up(3*e1 + 2*e2) ^ up(3*e1 - 2*e2) ^ e1
circle = up(e1) ^ up(-e1 + 2*e2) ^ up(-e1 - 2*e2)
```

We'll create copies of the point and line reflected in the circle, using  $X = C\hat{X}\tilde{C}$ , where  $\hat{X}$  is the grade involution.

```
[3]: point_refl = circle * point.gradeInvol() * ~circle
line_refl = circle * line.gradeInvol() * ~circle
```

## pyganja

pyganja is a python interface to the ganja.js ([github](#)) library. To use it, typically we need to import two names from the library:

```
[4]: from pyganja import GanjaScene, draw
import pyganja; pyganja.__version__

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
gui, cef functions will be unavailable
from .script_api import *

[4]: '0.0.14'
```

GanjaScene lets us build scenes out of geometric objects, with attached labels and RGB colors:

```
[5]: sc = GanjaScene()
sc.add_object(point, color=(255, 0, 0), label='point')
sc.add_object(line, color=(0, 255, 0), label='line')
sc.add_object(circle, color=(0, 0, 255), label='circle')

[6]: sc_refl = GanjaScene()
sc_refl.add_object(point_refl, color=(128, 0, 0), label='point_refl')
sc_refl.add_object(line_refl, color=(0, 128, 0), label='line_refl')
```

Once we've built our scene, we can draw it, specifying a scale (which here we use to zoom out), and the signature of our algebra (which defaults to conformal 3D):

```
[7]: draw(sc, sig=layout.sig, scale=0.5)

<IPython.core.display.Javascript object>
```

A cool feature of GanjaScene is the ability to use + to draw both scenes together:

```
[8]: draw(sc + sc_refl, sig=layout.sig, scale=0.5)

<IPython.core.display.Javascript object>
```

### `mpl_toolkits.clifford`

While `ganja.js` produces great diagrams, it's hard to combine them with other plotting tools. `mpl_toolkits.clifford` works within `matplotlib`.

```
[9]: from matplotlib import pyplot as plt
plt.ioff() # we'll ask for plotting when we want it

# if you're editing this locally, you'll get an interactive UI if you uncomment the_
↳following
#
# %matplotlib notebook

from mpl_toolkits.clifford import plot
import mpl_toolkits.clifford; mpl_toolkits.clifford.__version__

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳python3.8/site-packages/traitlets/traitlets.py:3030: FutureWarning: --rc={'figure.
↳dpi': 96, 'savefig.transparent': True} for dict-traits is deprecated in traitlets 5.
↳0. You can pass --rc <key=value> ... multiple times to add items to a dict.
warn(

[9]: '0.0.3'
```

Assembling the plot is a lot more work, but we also get much more control:

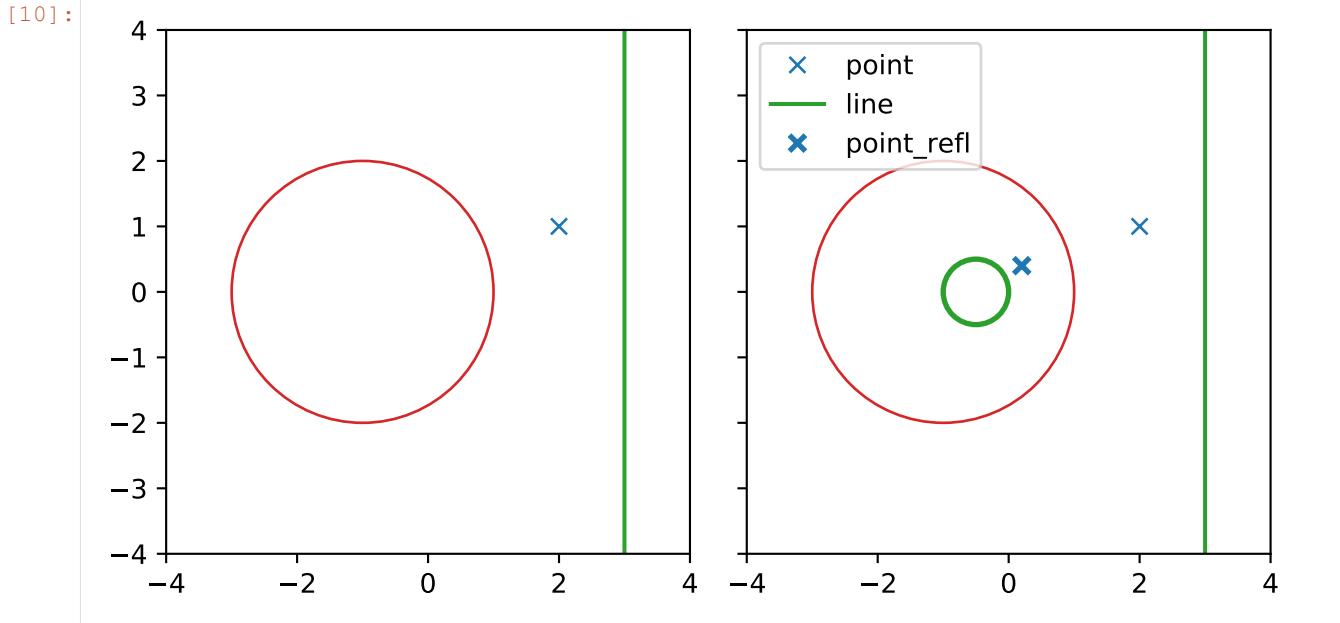
```
[10]: # standard matplotlib stuff - construct empty plots side-by-side, and set the scaling
fig, (ax_before, ax_both) = plt.subplots(1, 2, sharex=True, sharey=True)
ax_before.set(xlim=[-4, 4], ylim=[-4, 4], aspect='equal')
ax_both.set(xlim=[-4, 4], ylim=[-4, 4], aspect='equal')

# plot the objects before reflection on both plots
for ax in (ax_before, ax_both):
    plot(ax, [point], color='tab:blue', label='point', marker='x', linestyle=' ')
    plot(ax, [line], color='tab:green', label='line')
    plot(ax, [circle], color='tab:red', label='circle')

# plot the objects after reflection, with thicker lines
plot(ax_both, [point_refl], color='tab:blue', label='point_refl', marker='x',
↳linestyle=' ', markeredgewidth=2)
plot(ax_both, [line_refl], color='tab:green', label='line_refl', linewidth=2)

fig.tight_layout()
ax_both.legend()

# show the figure
fig
```



### G3C

Let's repeat the above, but with 3D Conformal Geometric Algebra. Note that if you're viewing these docs in a jupyter notebook, the lines below will replace all your 2d variables with 3d ones

```
[11]: from clifford.g3c import *

[12]: point = up(2*e1+e2)
line = up(3*e1 + 2*e2) ^ up(3*e1 - 2*e2) ^ e1f
circle = up(e1) ^ up(-e1 + 1.6*e2 + 1.2*e3) ^ up(-e1 - 1.6*e2 - 1.2*e3)
sphere = up(3*e1) ^ up(e1) ^ up(2*e1 + e2) ^ up(2*e1 + e3)

[13]: # note that due to floating point rounding, we need to truncate back to a single_
↪grade here, with ``grade``
point_refl = homo((circle * point.gradeInvol() * ~circle)(1))
line_refl = (circle * line.gradeInvol() * ~circle)(3)
sphere_refl = (circle * sphere.gradeInvol() * ~circle)(4)
```

### pyganja

Once again, we can create a pair of scenes exactly as before

```
[14]: sc = GanjaScene()
sc.add_object(point, color=(255, 0, 0), label='point')
sc.add_object(line, color=(0, 255, 0), label='line')
sc.add_object(circle, color=(0, 0, 255), label='circle')
sc.add_object(sphere, color=(0, 255, 255), label='sphere')

[15]: sc_refl = GanjaScene()
sc_refl.add_object(point_refl, color=(128, 0, 0), label='point_refl')
```

(continues on next page)

(continued from previous page)

```
sc_refl.add_object(line_refl.normal(), color=(0, 128, 0), label='line_refl')
sc_refl.add_object(sphere_refl.normal(), color=(0, 128, 128), label='sphere_refl')
```

But this time, when we draw them we don't need to pass `sig`. Better yet, we can rotate the 3D world around using left click, pan with right click, and zoom with the scroll wheel.

```
[16]: draw(sc + sc_refl, scale=0.5)

<IPython.core.display.Javascript object>
```

Some more example of using `pyganja` to visualize 3D CGA can be found in the [interpolation](#) and [clustering](#) notebooks.

### `mpl_toolkits.clifford`

The 3D approach for `matplotlib` is much the same. Note that due to poor handling of rounding errors in `clifford.tools.classify`, a call to `.normal()` is needed. Along with explicit grade selection, this is a useful trick to try and get something to render which otherwise would not.

```
[17]: # standard matplotlib stuff - construct empty plots side-by-side, and set the scaling
fig, (ax_before, ax_both) = plt.subplots(1, 2, subplot_kw=dict(projection='3d'),
    ↳ figsize=(8, 4))
ax_before.set(xlim=[-4, 4], ylim=[-4, 4], zlim=[-4, 4])
ax_both.set(xlim=[-4, 4], ylim=[-4, 4], zlim=[-4, 4])

# plot the objects before reflection on both plots
for ax in (ax_before, ax_both):
    plot(ax, [point], color='tab:red', label='point', marker='x', linestyle=' ')
    plot(ax, [line], color='tab:green', label='line')
    plot(ax, [circle], color='tab:blue', label='circle')
    plot(ax, [sphere], color='tab:cyan') # labels do not work for spheres: pygae/mpl_
    ↳ toolkits.clifford#5

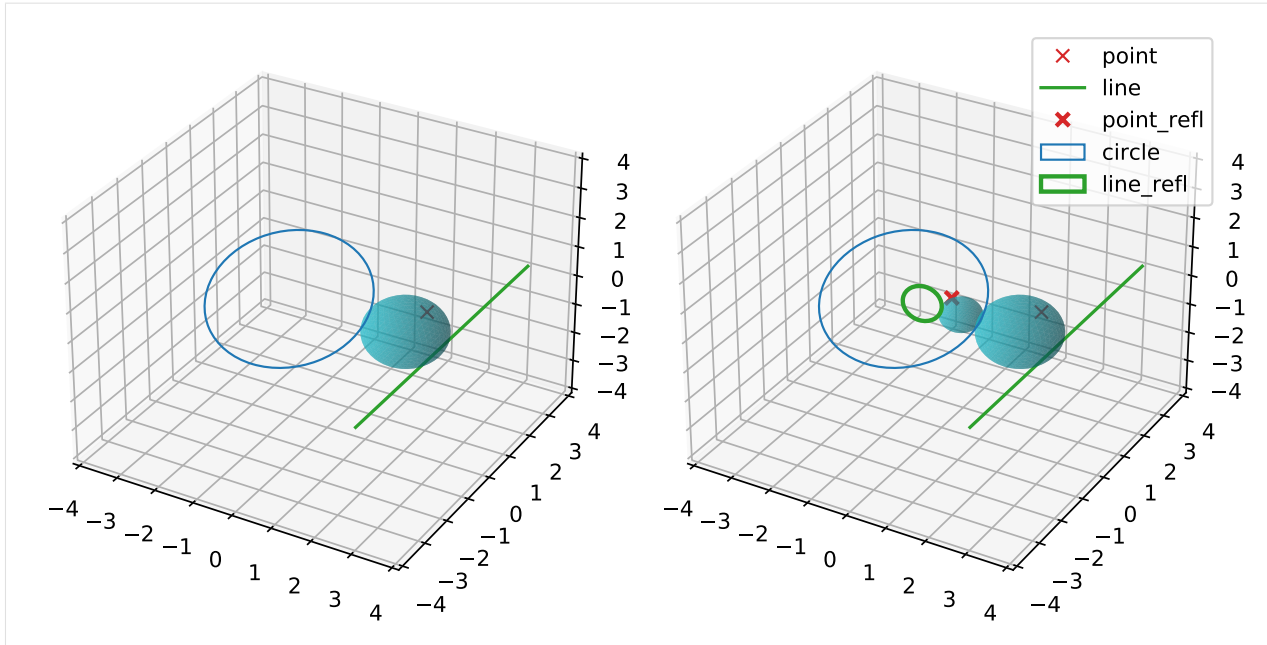
# plot the objects after reflection
plot(ax_both, [point_refl], color='tab:red', label='point_refl', marker='x',
    ↳ linestyle=' ', markeredgewidth=2)
plot(ax_both, [line_refl.normal()], color='tab:green', label='line_refl', linewidth=2)
plot(ax_both, [sphere_refl], color='tab:cyan')

fig.tight_layout()
ax_both.legend()

# show the figure
fig
```



[17]:



Some more example of using `mpl_toolkits.clifford` to visualize 3D CGA can be found in the examples folder of the `mpl_toolkits.clifford` repository, [here](#).

..... doc/tutorials/cga/visualization-tools.ipynb ends here.

The following section was generated from docs/tutorials/cga/object-oriented.ipynb .....

### 12.3.2 Object Oriented CGA

This is a shelled out demo for a object-oriented approach to CGA with `clifford`. The CGA object holds the original layout for an arbitrary geometric algebra, and the conformatized version. It provides up/down projections, as well as easy ways to generate objects and operators.

#### Quick Use Demo

```
[1]: from clifford.cga import CGA, Round, Translation
from clifford import Cl

g3, blades = Cl(3)

cga = CGA(g3) # make cga from existing ga
# or
cga = CGA(3) # generate cga from dimension of 'base space'

locals().update(cga.blades) # put ga's blades in local namespace

C = cga.round(e1, e2, e3, -e2) # generate unit sphere from points
C

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
gui, cef functions will be unavailable
from .script_api import *
```

```
[1]: Sphere
```

```
[2]: ## Objects
cga.round()           # from None
cga.round(3)          # from dim of space
cga.round(e1,e2,e3,-e2) # from points
cga.round(e1,e2,e3)    # from points
cga.round(e1,e2)       # from points
cga.round((e1,3))      # from center, radius
cga.round(cga.round().mv) # from existing multivector

cga.flat()            # from None
cga.flat(2)           # from dim of space
cga.flat(e1,e2)       # from points
cga.flat(cga.flat().mv) # from existing multivector

## Operations
cga.dilation()        # from from None
cga.dilation(.4)      # from int

cga.translation()     # from None
cga.translation(e1+e2) # from vector
cga.translation(cga.down(cga.null_vector()))

cga.rotation()        # from None
cga.rotation(e12+e23) # from bivector

cga.transversion(e1+e2).mv
```

```
/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳python3.8/site-packages/clifford/_multivector.py:263: RuntimeWarning: divide by_
↳zero encountered in true_divide
    newValue = self.value / other
/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳python3.8/site-packages/clifford/_multivector.py:263: RuntimeWarning: invalid value_
↳encountered in true_divide
    newValue = self.value / other
```

```
[2]: 1.0 + (0.5^e14) - (0.5^e15) + (0.5^e24) - (0.5^e25)
```

```
[3]: cga.round().inverted()
```

```
[3]: (0.87541^e1234) - (0.40372^e1235) + (0.29621^e1245) + (1.22355^e1345) + (0.13618^
↳e2345)
```

```
[4]: D = cga.dilation(5)
cga.down(D(e1))
```

```
[4]: (5.0^e1)
```

```
[5]: C.mv # any CGA object/operator has a multivector
```

```
[5]: (1.0^e1235)
```

```
[6]: C.center_down,C.radius # some properties of spheres
```

```
[6]: (0, 1.0)
```

```
[7]: T = cga.translation(e1+e2) # make a translation
     C_ = T(C)                 # translate the sphere
     cga.down(C_.center)       # compute center again
```

```
[7]: (1.0^e1) + (1.0^e2)
```

```
[8]: cga.round()             # no args == random sphere
     cga.translation()       # random translation
```

```
[8]: Translation
```

```
[9]: if 1 in map(int, [1,2]):
     print(3)
```

```
3
```

## Objects

### Vectors

```
[10]: a = cga.base_vector() # random vector with components in base space only
      a
```

```
[10]: -(1.25748^e1) + (1.1739^e2) - (1.23633^e3)
```

```
[11]: cga.up(a)
```

```
[11]: -(1.25748^e1) + (1.1739^e2) - (1.23633^e3) + (1.7439^e4) + (2.7439^e5)
```

```
[12]: cga.null_vector() # create null vector directly
```

```
[12]: (0.19804^e1) - (0.40804^e2) + (1.68569^e3) + (1.02364^e4) + (2.02364^e5)
```

### Sphere (point pair, circles)

```
[13]: C = cga.round(e1, e2, -e1, e3) # generates sphere from points
      C = cga.round(e1, e2, -e1)     # generates circle from points
      C = cga.round(e1, e2)          # generates point-pair from points
      #or
      C2 = cga.round(2)               # random 2-sphere (sphere)
      C1 = cga.round(1)               # random 1-sphere, (circle)
      C0 = cga.round(0)               # random 0-sphere, (point pair)
```

```
C1.mv # access the multivector
```

```
[13]: (0.04822^e123) - (0.10071^e124) - (0.06379^e125) - (0.22046^e134) - (0.34907^e135) +
      ↪ (0.4374^e145) + (0.1968^e234) + (0.41748^e235) - (0.61158^e245) - (0.48399^e345)
```

```
[14]: C = cga.round(e1, e2, -e1, e3)
      C.center, C.radius # spheres have properties
```

```
[14]:  $(-(1.0^4) + (1.0^5), 1.0)$ 

[15]: cga.down(C.center) == C.center_down
[15]: True

[16]: C_ = cga.round().from_center_radius(C.center,C.radius)
C_.center,C_.radius
[16]:  $(-(2.0^4) + (2.0^5), 0.7071067811865476)$ 
```

## Operators

```
[17]: T = cga.translation(e1) # generate translation
T.mv
[17]:  $1.0 - (0.5^{14}) - (0.5^{15})$ 

[18]: C = cga.round(e1, e2, -e1)
T.mv*C.mv**T.mv # translate a sphere
[18]:  $-(0.5^{124}) + (0.5^{125}) - (1.0^{245})$ 

[19]: T(C) # shorthand call, same as above. returns type of arg
[19]: Circle

[20]: T(C).center
[20]:  $(2.0^1) + (2.0^5)$ 

[ ]:

[ ]:
..... doc/tutorials/cga/object-oriented.ipynb ends here.
```

The following section was generated from docs/tutorials/cga/interpolation.ipynb .....

### 12.3.3 Example 1 Interpolating Conformal Objects

In this example we will look at a few of the tools provided by the clifford package for (4,1) conformal geometric algebra (CGA) and see how we can use them in a practical setting to interpolate geometric primitives.

The first step in using the package for CGA is to generate and import the algebra:

```
[1]: from clifford.g3c import *
blades

[1]: {': 1,
'e1': (1^e1),
'e2': (1^e2),
'e3': (1^e3),
'e4': (1^e4),
'e5': (1^e5),
'e12': (1^e12),
```

(continues on next page)

(continued from previous page)

```
'e13': (1^e13),
'e14': (1^e14),
'e15': (1^e15),
'e23': (1^e23),
'e24': (1^e24),
'e25': (1^e25),
'e34': (1^e34),
'e35': (1^e35),
'e45': (1^e45),
'e123': (1^e123),
'e124': (1^e124),
'e125': (1^e125),
'e134': (1^e134),
'e135': (1^e135),
'e145': (1^e145),
'e234': (1^e234),
'e235': (1^e235),
'e245': (1^e245),
'e345': (1^e345),
'e1234': (1^e1234),
'e1235': (1^e1235),
'e1245': (1^e1245),
'e1345': (1^e1345),
'e2345': (1^e2345),
'e12345': (1^e12345) }
```

This creates an algebra with the required signature and imports the basis blades into the current workspace. We can check our metric by squaring our grade 1 multivectors.

```
[2]: print('e1*e1 ', e1*e1)
      print('e2*e2 ', e2*e2)
      print('e3*e3 ', e3*e3)
      print('e4*e4 ', e4*e4)
      print('e5*e5 ', e5*e5)
```

```
e1*e1  1
e2*e2  1
e3*e3  1
e4*e4  1
e5*e5 -1
```

As expected this gives us 4 basis vectors that square to 1.0 and one that squares to -1.0, therefore confirming our metric is (4,1).

The `up()` function implements the mapping of vectors from standard 3D space to conformal space. We can use this to construct conformal objects to play around with.

For example a line at the origin:

```
[3]: line_a = ( up(0)^up(e1+e2)^einf ).normal()
      print(line_a)
```

```
(0.70711^e145) + (0.70711^e245)
```

The `tools` submodule of the `clifford` package contains a wide array of algorithms and tools that can be useful for manipulating objects in CGA. We will use these tools to generate rotors that rotate and translate objects:

```
[4]: from clifford.tools.g3 import *
from clifford.tools.g3c import *
from numpy import pi

rotation_radians = pi/4
euc_vector_in_plane_m = e1
euc_vector_in_plane_n = e3

euc_translation = -5.2*e1 + 3*e2 - pi*e3

rotor_rotation = generate_rotation_rotor(rotation_radians, euc_vector_in_plane_m, euc_
↳vector_in_plane_n)
rotor_translation = generate_translation_rotor(euc_translation)
print(rotor_rotation)
print(rotor_translation)

combined_rotor = (rotor_translation*rotor_rotation).normal()

line_b = (combined_rotor*line_a*~combined_rotor).normal()
print(line_b)

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
↳gui, cef functions will be unavailable
from .script_api import *

0.92388 - (0.38268^e13)
1.0 + (2.6^e14) + (2.6^e15) - (1.5^e24) - (1.5^e25) + (1.5708^e34) + (1.5708^e35)
- (5.17696^e124) - (5.17696^e125) - (1.0292^e134) - (1.0292^e135) + (0.5^e145) + (3.
↳72144^e234) + (3.72144^e235) + (0.70711^e245) + (0.5^e345)
```

In the above snippet of code we have generated rotors for translation and rotation, then combined these, then applied the combined rotor to the original line that we made.

## Visualizations

The `clifford` package can be used alongside `pyganja` to render CGA objects which can be rotated interactively in a jupyter notebook:

```
[5]: from pyganja import GanjaScene, draw
sc = GanjaScene()
sc.add_object(line_a,color=0xFF0000, label='a')
sc.add_object(line_b,color=0x00FF00, label='b')
draw(sc, scale=0.1)

<IPython.core.display.Javascript object>
```

We can also interpolate the objects using the tools in `clifford` and can visualise the result

```
[6]: def interpolate_objects_linearly(L1, L2, n_steps=10, color_1=np.array([255,0,0]),
↳color_2=np.array([0,255,0])):
    alpha_list = np.linspace(0, 1, num=n_steps)
    intermediary_list = []
    sc = GanjaScene()
    for alpha in alpha_list:
        intermediate_color = (alpha*color_1 + (1-alpha)*color_2).astype(np.uint8)
        intermediate_object = interp_objects_root(L1, L2, alpha)
```

(continues on next page)

(continued from previous page)

```

intermediary_list.append(intermediate_object)
color_string = int(
    (intermediate_color[0] << 16) | (intermediate_color[1] << 8) |
↪intermediate_color[2]
)
sc.add_object(intermediate_object, color_string)
return intermediary_list, sc

```

```

[7]: intermediary_list, finished_scene = interpolate_objects_linearly(line_a, line_b)
draw(finished_scene, scale=0.1)

<IPython.core.display.Javascript object>

```

We can do the same for all the other geometric primitives as well

### Circles

```

[8]: circle_a = (up(0)^up(e1)^up(e2)).normal()
circle_b = (combined_rotor*circle_a*~combined_rotor).normal()
intermediary_list, finished_scene = interpolate_objects_linearly(circle_a, circle_b)
draw(finished_scene, scale=0.1)

<IPython.core.display.Javascript object>

```

### Point pairs

```

[9]: point_pair_a = (up(e3)^up(e1+e2)).normal()
point_pair_b = (combined_rotor*point_pair_a*~combined_rotor).normal()
intermediary_list, finished_scene = interpolate_objects_linearly(point_pair_a, point_
↪pair_b)
draw(finished_scene, scale=0.1)

<IPython.core.display.Javascript object>

```

### Planes

```

[10]: plane_a = (up(0)^up(e1)^up(e2)^einf).normal()
plane_b = (combined_rotor*plane_a*~combined_rotor).normal()
intermediary_list, finished_scene = interpolate_objects_linearly(plane_a, plane_b)
draw(finished_scene)

<IPython.core.display.Javascript object>

```

## Spheres

```
[11]: sphere_a = (up(0)^up(e1)^up(e2)^up(e3)).normal()
sphere_b = (combined_rotor*sphere_a*~combined_rotor).normal()
intermediary_list, finished_scene = interpolate_objects_linearly(sphere_a, sphere_b)
draw(finished_scene, scale=0.1)

<IPython.core.display.Javascript object>
..... doc/tutorials/cga/interpolation.ipynb ends here.
```

The following section was generated from docs/tutorials/cga/clustering.ipynb .....

### 12.3.4 Example 2 Clustering Geometric Objects

In this example we will look at a few of the tools provided by the clifford package for (4,1) conformal geometric algebra (CGA) and see how we can use them in a practical setting to cluster geometric objects via the simple K-means clustering algorithm provided in clifford.tools

As before the first step in using the package for CGA is to generate and import the algebra:

```
[1]: from clifford.g3c import *
print('e1*e1 ', e1*e1)
print('e2*e2 ', e2*e2)
print('e3*e3 ', e3*e3)
print('e4*e4 ', e4*e4)
print('e5*e5 ', e5*e5)

e1*e1  1
e2*e2  1
e3*e3  1
e4*e4  1
e5*e5 -1
```

The tools submodule of the clifford package contains a wide array of algorithms and tools that can be useful for manipulating objects in CGA. In this case we will be generating a large number of objects and then segmenting them into clusters.

We first need an algorithm for generating a cluster of objects in space. We will construct this cluster by generating a random object and then repeatedly disturbing this object by some small fixed amount and storing the result:

```
[2]: from clifford.tools.g3c import *
import numpy as np

def generate_random_object_cluster(n_objects, object_generator, max_cluster_trans=1.0,
    ↪ max_cluster_rot=np.pi/8):
    """ Creates a cluster of random objects """
    ref_obj = object_generator()
    cluster_objects = []
    for i in range(n_objects):
        r = random_rotation_translation_rotor(maximum_translation=max_cluster_trans,
    ↪ maximum_angle=max_cluster_rot)
        new_obj = apply_rotor(ref_obj, r)
        cluster_objects.append(new_obj)
    return cluster_objects
```



```
/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
↳gui, cef functions will be unavailable
from .script_api import *
```

We can use this function to create a cluster and then we can visualise this cluster with `pyganja`.

```
[3]: from pyganja import *
clustered_circles = generate_random_object_cluster(10, random_circle)
sc = GanjaScene()
for c in clustered_circles:
    sc.add_object(c, rgb2hex([255,0,0]))
draw(sc, scale=0.05)

<IPython.core.display.Javascript object>
```

This cluster generation function appears in clifford tools by default and it can be imported as follows:

```
[4]: from clifford.tools.g3c import generate_random_object_cluster
```

Now that we can generate individual clusters we would like to generate many:

```
[5]: def generate_n_clusters( object_generator, n_clusters, n_objects_per_cluster ):
    object_clusters = []
    for i in range(n_clusters):
        cluster_objects = generate_random_object_cluster(n_objects_per_cluster,
↳object_generator,
                                                                    max_cluster_trans=0.5, max_
↳cluster_rot=np.pi / 16)
        object_clusters.append(cluster_objects)
    all_objects = [item for sublist in object_clusters for item in sublist]
    return all_objects, object_clusters
```

Again this function appears by default in clifford tools and we can easily visualise the result:

```
[6]: from clifford.tools.g3c import generate_n_clusters

all_objects, object_clusters = generate_n_clusters(random_circle, 2, 5)
sc = GanjaScene()
for c in all_objects:
    sc.add_object(c, rgb2hex([255,0,0]))
draw(sc, scale=0.05)

<IPython.core.display.Javascript object>
```

Given that we can now generate multiple clusters of objects we can test algorithms for segmenting them.

The function `run_n_clusters` below generates a lot of objects distributed into `n` clusters and then attempts to segment the objects to recover the clusters.

```
[7]: from clifford.tools.g3c.object_clustering import n_clusters_objects
import time

def run_n_clusters( object_generator, n_clusters, n_objects_per_cluster, n_
↳shotgunning):
    all_objects, object_clusters = generate_n_clusters( object_generator, n_clusters,
↳n_objects_per_cluster )
    [new_labels, centroids, start_labels, start_centroids] = n_clusters_objects(n_
↳clusters, all_objects,
```

(continues on next page)

(continued from previous page)

```

↪initial_centroids=None,
                                n_
↪shotgunning=n_shotgunning,
                                ↪
↪averaging_method='unweighted')
    return all_objects, new_labels, centroids

```

Lets try it!

```

[8]: def visualise_n_clusters(all_objects, centroids, labels,
                               color_1=np.array([255, 0, 0]),
                               color_2=np.array([0, 255, 0])):
    """
    Utility method for visualising several clusters and their respective centroids
    using pyganja
    """
    alpha_list = np.linspace(0, 1, num=len(centroids))
    sc = GanjaScene()
    for ind, this_obj in enumerate(all_objects):
        alpha = alpha_list[labels[ind]]
        cluster_color = (alpha * color_1 + (1 - alpha) * color_2).astype(np.int)
        sc.add_object(this_obj, rgb2hex(cluster_color))

    for c in centroids:
        sc.add_object(c, Color.BLACK)

    return sc

```

```
object_generator = random_circle
```

```

n_clusters = 3
n_objects_per_cluster = 10
n_shotgunning = 60
all_objects, labels, centroids = run_n_clusters(object_generator, n_clusters,
                                                n_objects_per_cluster, n_
↪shotgunning)

```

```

sc = visualise_n_clusters(all_objects, centroids, labels,
                           color_1=np.array([255, 0, 0]),
                           color_2=np.array([0, 255, 0]))
draw(sc, scale=0.05)

```

```

<ipython-input-1-a00e05e6e43b>:12: DeprecationWarning: `np.int` is a deprecated alias
↪for the builtin `int`. To silence this warning, use `int` by itself. Doing this
↪will not modify any behavior and is safe. When replacing `np.int`, you may wish to
↪use e.g. `np.int64` or `np.int32` to specify the precision. If you wish to review
↪your current use, check the release note link for additional information.
Deprecated in NumPy 1.20; for more details and guidance: https://numpy.org/devdocs/
↪release/1.20.0-notes.html#deprecations
    cluster_color = (alpha * color_1 + (1 - alpha) * color_2).astype(np.int)

```

```
<IPython.core.display.Javascript object>
```

..... doc/tutorials/cga/clustering.ipynb ends here.

The following section was generated from docs/tutorials/cga/robotic-manipulators.ipynb .....

## 12.3.5 Application to Robotic Manipulators

This notebook is intended to expand upon the ideas in part of the presentation [Robots, Ganja & Screw Theory](#)

### Serial manipulator

(slides)

Let's consider a 2-link 3 DOF arm. We'll model the links within the robot with rotors, which transform to the coordinate frame of the end of each link. This is very similar to the approach that would classically be taken with 4x4 matrices.

We're going to define our class piecewise as we go along here. To aid that, we'll write a simple base class to let us do just that. In your own code, there's no need to do this.

```
[1]: class AddMethodsAsWeGo:
    @classmethod
    def __add_method(cls, m):
        if isinstance(m, property):
            name = (m.fget or m.fset).__name__
        else:
            name = m.__name__
        setattr(cls, name, m)
```

Let's start by defining some names for the links, and a place to store our parameters:

```
[2]: from enum import Enum

class Links(Enum):
    BASE = 'b'
    SHOULDER = 's'
    UPPER = 'u'
    ELBOW = 'e'
    FOREARM = 'f'
    ENDPOINT = 'n'

class SerialRobot(AddMethodsAsWeGo):
    def __init__(self, rho, l):
        self.l = l
        self.rho = rho
        self._thetas = (0, 0, 0)

    @property
    def thetas(self):
        return self._thetas
```

## Forward kinematics

(slides)

As a reminder, we can construct rotation and translation motors as:

$$\begin{aligned}
 T(a) &= \exp\left(\frac{1}{2}n_{\infty} \wedge a\right) \\
 &= 1 + \frac{1}{2}n_{\infty}a \\
 R(\theta, \hat{B}) &= \exp\left(\frac{1}{2}\theta\hat{B}\right) \\
 &= \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{B}
 \end{aligned} \tag{12.1}$$

Applying these to our geometry, we get

$$\begin{aligned}
 R_{\text{base} \leftarrow \text{shoulder}} &= R(\theta_0, e_1 \wedge e_3) \\
 R_{\text{shoulder} \leftarrow \text{upper arm}} &= R(\theta_1, e_1 \wedge e_2) \\
 R_{\text{upper arm} \leftarrow \text{elbow}} &= T(\rho e_1) \\
 R_{\text{elbow} \leftarrow \text{forearm}} &= R(\theta_2, e_1 \wedge e_2) \\
 R_{\text{forearm} \leftarrow \text{endpoint}} &= T(\rho e_1)
 \end{aligned} \tag{12.5}$$

From which we can get the overall rotor to the frame of the endpoint, and the positions  $X$  and  $Y$ :

$$R_{\text{base} \leftarrow \text{elbow}} = R_{\text{base} \leftarrow \text{shoulder}} R_{\text{shoulder} \leftarrow \text{upper arm}} R_{\text{upper arm} \leftarrow \text{elbow}} \tag{12.11}$$

$$X = R_{\text{base} \leftarrow \text{elbow}} n_0 \tilde{R}_{\text{base} \leftarrow \text{elbow}} \tag{12.12}$$

$$R_{\text{base} \leftarrow \text{endpoint}} = R_{\text{base} \leftarrow \text{shoulder}} R_{\text{shoulder} \leftarrow \text{upper arm}} R_{\text{upper arm} \leftarrow \text{elbow}} R_{\text{elbow} \leftarrow \text{forearm}} R_{\text{forearm} \leftarrow \text{endpoint}} \tag{12.13}$$

$$Y = R_{\text{base} \leftarrow \text{endpoint}} n_0 \tilde{R}_{\text{base} \leftarrow \text{endpoint}} \tag{12.14}$$

(12.15)

We can write this as:

```
[3]: from clifford.g3c import *
from clifford.tools.g3c import generate_translation_rotor, apply_rotor
from clifford.tools.g3 import generate_rotation_rotor

def _update_chain(rotors, a, b, c):
    rotors[a, c] = rotors[a, b] * rotors[b, c]

@SerialRobot._add_method
@SerialRobot.thetas.setter
def thetas(self, value):
    theta0, theta1, theta2 = self._thetas = value
    # shorthands for brevity
    R = generate_rotation_rotor
    T = generate_translation_rotor

    rotors = {}
    rotors[Links.BASE, Links.SHOULDER] = R(theta0, e1, e3)
    rotors[Links.SHOULDER, Links.UPPER] = R(theta1, e1, e2)
    rotors[Links.UPPER, Links.ELBOW] = T(self.rho * e1)
    rotors[Links.ELBOW, Links.FOREARM] = R(theta2, e1, e2)
```

(continues on next page)

(continued from previous page)

```

rotors[Links.FOREARM, Links.ENDPOINT] = T(-self.l * e1)

_update_chain(rotors, Links.BASE, Links.SHOULDER, Links.UPPER)
_update_chain(rotors, Links.BASE, Links.UPPER, Links.ELBOW)
_update_chain(rotors, Links.BASE, Links.ELBOW, Links.FOREARM)
_update_chain(rotors, Links.BASE, Links.FOREARM, Links.ENDPOINT)
self.rotors = rotors

@SerialRobot._add_method
@property
def y_pos(self):
    return apply_rotor(eo, self.rotors[Links.BASE, Links.ENDPOINT])

@SerialRobot._add_method
@property
def x_pos(self):
    return apply_rotor(eo, self.rotors[Links.BASE, Links.ELBOW])

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
gui, cef functions will be unavailable
from .script_api import *
```

Let's write a renderer so we can check this all works

```

[4]: from pyganja import GanjaScene

def add_rotor(sc: GanjaScene, r, *, label=None, color=None, scale=0.1):
    """ show how a rotor transforms the axes at the origin """
    y = apply_rotor(eo, r)
    y_frame = [
        apply_rotor(d, r)
        for d in [up(scale*e1), up(scale*e2), up(scale*e3)]
    ]
    sc.add_object(y, label=label, color=color)
    sc.add_facet([y, y_frame[0]], color=(255, 0, 0))
    sc.add_facet([y, y_frame[1]], color=(0, 255, 0))
    sc.add_facet([y, y_frame[2]], color=(0, 0, 255))

@SerialRobot._add_method
def to_scene(self):
    sc = GanjaScene()
    axis_scale = 0.1
    link_scale = 0.05
    arm_color = (192, 192, 192)

    base_obj = (up(0.2*e1)^up(0.2*e3)^up(-0.2*e1)).normal()
    sc.add_object(base_obj, color=0)

    shoulder_axis = [
        apply_rotor(p, self.rotors[Links.BASE, Links.UPPER])
        for p in [up(axis_scale*e3), up(-axis_scale*e3)]
    ]
    sc.add_facet(shoulder_axis, color=(0, 0, 128))
    shoulder_angle = [
```

(continues on next page)

(continued from previous page)

```

        apply_rotor(eo, self.rotors[Links.BASE, Links.SHOULDER]),
        apply_rotor(up(axis_scale*e1), self.rotors[Links.BASE, Links.SHOULDER]),
        apply_rotor(up(axis_scale*e1), self.rotors[Links.BASE, Links.UPPER]),
    ]
    sc.add_facet(shoulder_angle, color=(0, 0, 128))

    upper_arm_points = [
        apply_rotor(up(link_scale*e3), self.rotors[Links.BASE, Links.UPPER]),
        apply_rotor(up(-link_scale*e3), self.rotors[Links.BASE, Links.UPPER]),
        apply_rotor(up(link_scale*e3), self.rotors[Links.BASE, Links.ELBOW]),
        apply_rotor(up(-link_scale*e3), self.rotors[Links.BASE, Links.ELBOW])
    ]
    sc.add_facet(upper_arm_points[:3], color=arm_color)
    sc.add_facet(upper_arm_points[1:], color=arm_color)

    elbow_axis = [
        apply_rotor(p, self.rotors[Links.BASE, Links.ELBOW])
        for p in [up(axis_scale*e3), up(-axis_scale*e3)]
    ]
    sc.add_facet(elbow_axis, color=(0, 0, 128))

    forearm_points = [
        apply_rotor(up(link_scale*e3), self.rotors[Links.BASE, Links.FOREARM]),
        apply_rotor(up(-link_scale*e3), self.rotors[Links.BASE, Links.FOREARM]),
        apply_rotor(up(link_scale*e3), self.rotors[Links.BASE, Links.ENDPOINT]),
        apply_rotor(up(-link_scale*e3), self.rotors[Links.BASE, Links.ENDPOINT])
    ]
    sc.add_facet(forearm_points[:3], color=arm_color)
    sc.add_facet(forearm_points[1:], color=arm_color)

    add_rotor(sc, self.rotors[Links.BASE, Links.ELBOW], label='x', color=(128, 128, 128))
    add_rotor(sc, self.rotors[Links.BASE, Links.ENDPOINT], label='y', color=(128, 128, 128))

    return sc

```

We can now instantiate our robot

```
[5]: serial_robot = SerialRobot(rho=1, l=0.5)
```

Choose a trajectory

```
[6]: import math
theta_traj = [
    (math.pi/6 + i*math.pi/12, math.pi/3 - math.pi/12*i, 3*math.pi/4)
    for i in range(3)
]
```

And plot the robot in each state, using `ipywidgets` ([docs](#)) to let us plot ganja side-by-side. Unfortunately, `pyganja` provides no mechanism to animate these plots from python.

---

This will not render side-by-side in the online clifford documentation, but will in a local notebook.

---

```
[7]: import ipywidgets
from IPython.display import Latex, display
from pyganja import draw

outputs = [
    ipywidgets.Output(layout=ipywidgets.Layout(flex='1'))
    for i in range(len(theta_traj))
]
for output, thetas in zip(outputs, theta_traj):
    with output:
        # interesting part here - run the forward kinematics, print the angles we used
        serial_robot.thetas = thetas
        display(Latex(r"$\theta_i = {:.2f}, {:.2f}, {:.2f}$".format(*thetas)))
        draw(serial_robot.to_scene(), scale=1.5)
ipywidgets.HBox(outputs)

HBox(children=(Output(layout=Layout(flex='1')), Output(layout=Layout(flex='1')),
↳Output(layout=Layout(flex='1'...
```

## Inverse kinematics

(slides)

For the forward kinematics, we didn't actually need conformal geometric algebra at all—PGA would have done just fine, as all we needed were rotations and translations. The inverse kinematics of a serial manipulator is where CGA provide some nice tricks.

There are three facts we know about the position  $X$ , each of which describes a constraint surface

- $X$  must lie on a sphere with radius  $l$  centered at  $Y$ , which can be written

$$S^* = Y - \frac{1}{2}l^2n_\infty$$

- $X$  must lie on a sphere with radius  $\rho$  centered at  $n_o$ , which can be written

$$S_{\text{base}}^* = n_o - \frac{1}{2}\rho^2n_\infty$$

- $X$  must lie on a plane through  $n_o$ ,  $e_3$ , and  $Y$ , which can be written

$$\Pi = n_o \wedge \text{up}(e_3) \wedge Y \wedge n_\infty$$

Note that  $\Pi = 0$  is possible iff  $Y = \text{up}(ke_3)$ .

For  $X$  to satisfy all three constraints, we have

$$S \wedge X = S_{\text{base}} \wedge X = \Pi \wedge X = 0 \quad (12.16)$$

$$X \wedge \underbrace{(S \vee S_{\text{base}} \vee \Pi)}_P = 0 \quad \text{If } \Pi \neq 0 \quad (12.17)$$

$$X \wedge \underbrace{(S \vee S_{\text{base}})}_C = 0 \quad \text{otherwise} \quad (12.18)$$

$$(12.19)$$

By looking at the grade of the term labelled  $P$ , we conclude it must be a point-pair—which tells us  $X$  must lie in one of two locations. Similarly,  $C$  must be a circle.

```
[8]: @SerialRobot._add_method
def _get_x_constraints_for(self, Y):
    """ Get the space containing all possible elbow positions """
    # strictly should be undual, but we don't have that in clifford
    S = (Y - 0.5*self.l**2*einf).dual()
    S_base = (eo - 0.5*self.rho**2*einf).dual()
    Pi = eo ^ up(e2) ^ Y ^ einf
    return S, S_base, Pi

@SerialRobot._add_method
def _get_x_positions_for(self, Y):
    """ Get the space containing all possible elbow positions """
    S, S_base, Pi = self._get_x_constraints_for(Y)
    if Pi == 0:
        # any solution on the circle is OK
        return S & S_base
    else:
        # there are just two solutions
        return S & S_base & Pi
```

From the pointpair  $P$  we can extract the two possible  $X$  locations with:

$$X = \left[ 1 \pm \frac{P}{\sqrt{P\tilde{P}}} \right] (P \cdot n_\infty)$$

To be considered a full solution to the inverse kinematics problem, we need to produce the angles  $\theta_0, \theta_1, \theta_2$ . We can do this as follows

```
[9]: @SerialRobot._add_method
@SerialRobot.y_pos.setter
def y_pos(self, Y):
    R = generate_rotation_rotor
    T = generate_translation_rotor

    rotors = {}
    rotors[Links.UPPER, Links.ELBOW] = T(self.rho * e1)
    rotors[Links.FOREARM, Links.ENDPOINT] = T(-self.l * e1)

    x_options = self._get_x_positions_for(Y)
    if x_options.grades == {3}:
        # no need to adjust the base angle
        theta_0 = self.thetas[0]
        rotors[Links.BASE, Links.SHOULDER] = self.rotors[Links.BASE, Links.SHOULDER]
        # remove the rotation from x, intersect it with the plane of the links
        x_options = x_options & (eo ^ up(e3) ^ up(e1) ^ einf)
    else:
        y_down = down(Y)
        theta0 = math.atan2(y_down[(3,)], y_down[(1,)])
        rotors[Links.BASE, Links.SHOULDER] = R(theta0, e1, e3)

        # remove the first rotor from x
        x_options = apply_rotor(x_options, ~rotors[Links.BASE, Links.SHOULDER])

    # project out one end of the point-pair
    x = (1 - x_options.normal()) * (x_options | einf)

    x_down = down(x)
```

(continues on next page)



(continued from previous page)

```

theta1 = math.atan2(x_down[(2,)], x_down[(1,)])
rotors[Links.SHOULDER, Links.UPPER] = R(theta1, e1, e2)

_update_chain(rotors, Links.BASE, Links.SHOULDER, Links.UPPER)
_update_chain(rotors, Links.BASE, Links.UPPER, Links.ELBOW)

# remove the second rotor
Y = apply_rotor(Y, ~rotors[Links.BASE, Links.ELBOW])
y_down = down(Y)

theta2 = math.atan2(-y_down[(2,)], -y_down[(1,)])
rotors[Links.ELBOW, Links.FOREARM] = R(theta2, e1, e2)
_update_chain(rotors, Links.BASE, Links.ELBOW, Links.FOREARM)
_update_chain(rotors, Links.BASE, Links.FOREARM, Links.ENDPOINT)

self._thetas = (theta0, theta1, theta2)
self.rotors = rotors

```

Define a trajectory again, this time with a scene to render it:

```

[10]: y_traj = [
        up(0.3*e3 + 0.8*e2 - 0.25*e1),
        up(0.6*e3 + 0.8*e2),
        up(0.9*e3 + 0.8*e2 + 0.25*e1)
    ]

expected_scene = GanjaScene()
expected_scene.add_facet(y_traj[0:2], color=(255, 128, 128))
expected_scene.add_facet(y_traj[1:3], color=(255, 128, 128))

```

And we can run the inverse kinematics by setting `serial_robot.y_pos`:

```

[11]: outputs = [
        ipywidgets.Output(layout=ipywidgets.Layout(flex='1'))
        for i in range(len(y_traj))
    ]
first = True
for output, y in zip(outputs, y_traj):
    with output:
        # interesting part here - run the reverse kinematics, print the angles we used
        serial_robot.y_pos = y
        display(Latex(r"$\theta_i = {:.2f}, {:.2f}, {:.2f}$".format(*serial_robot.
→thetas)))
        sc = serial_robot.to_scene()

        # Show the spheres we used to construct the solution
        sc += expected_scene
        if first:
            extra_scene = GanjaScene()
            S, S_base, Pi = serial_robot._get_x_constraints_for(y)
            extra_scene.add_object(S_base, label='S_base', color=(255, 255, 128))
            extra_scene.add_object(S, label='S', color=(255, 128, 128))
            extra_scene.add_object(Pi, label='Pi', color=(128, 255, 192, 128))
            sc += extra_scene
        draw(sc, scale=1.5)
        first = False
ipywidgets.HBox(outputs)

```

```
HBox(children=(Output(layout=Layout(flex='1')), Output(layout=Layout(flex='1')),  
↪Output(layout=Layout(flex='1'))...
```

## Parallel manipulators

For now, refer to the presentation

(slides)

## Inverse kinematics

(slides)

For now, refer to the presentation

## Forward kinematics

(slides)

For now, refer to the presentation

..... doc/tutorials/cga/robotic-manipulators.ipynb ends here.  
..... doc/tutorials/cga/index.ipynb ends here.

The following section was generated from docs/tutorials/linear-transformations.ipynb .....

## LINEAR TRANSFORMATIONS

When working in regular vector spaces, a common tool is a linear transformation, typically in the form of a matrix.

While geometric algebra already provides the rotors as a means of describing transformations (see [the CGA tutorial section](#)), there are types of linear transformation that are not suitable for this representation.

This tutorial leans heavily on the explanation of linear transformations in [DFM09], chapter 4. It explores the `clifford.transformations` submodule.

### 13.1 Vector transformations in linear algebra

As a brief reminder, we can represent transforms in  $\mathbb{R}^3$  using the matrices in  $\mathbb{R}^{3 \times 3}$ :

```
[1]: import numpy as np

rot_and_scale_x = np.array([
    [1, 0, 0],
    [0, 1, -1],
    [0, 1, 1],
])
```

We can read this as a table, where each column corresponds to a component of the input vector, and each row a component of the output:

```
[2]: def show_table(data, cols, rows):
    # trick to get a nice looking table in a notebook
    import pandas as pd; return pd.DataFrame(data, columns=cols, index=rows)
```

```
[3]: show_table(rot_and_scale_x, ["$\mathit{in}_s$" % c for c in "xyz"], ["$\mathit{out}_s$" % c for c in "xyz"])
```

```
[3]:
```

	$\mathit{in}_x$	$\mathit{in}_y$	$\mathit{in}_z$
$\mathit{out}_x$	1	0	0
$\mathit{out}_y$	0	1	-1
$\mathit{out}_z$	0	1	1

We can apply it to some vectors using the @ matrix multiply operator:

```
[4]: v1 = np.array([1, 0, 0])
v2 = np.array([0, 1, 0])
v3 = np.array([0, 0, 1])

(
```

(continues on next page)

(continued from previous page)

```

    rot_and_scale_x @ v1,
    rot_and_scale_x @ v2,
    rot_and_scale_x @ v3,
)

```

```
[4]: (array([1, 0, 0]), array([0, 1, 1]), array([ 0, -1, 1]))
```

We say this transformation is linear because  $f(a + b) = f(a) + f(b)$ :

```
[5]: assert np.array_equal(
    rot_and_scale_x @ (2*v1 + 3*v2),
    2 * (rot_and_scale_x @ v1) + 3 * (rot_and_scale_x @ v2)
)
```

## 13.2 Multivector transformations in geometric algebra

How would we go about applying `rot_and_scale_x` in a geometric algebra? Clearly we can apply it to vectors in the same way as before, which we can do by unpacking coefficients and repacking them:

```
[6]: from clifford.g3 import *

v = 2*e1 + 3*e2
v_trans = layout.MultiVector()
v_trans[1,], v_trans[2,], v_trans[3,] = rot_and_scale_x @ [v[1,], v[2,], v[3,]]
v_trans

[6]: (2.0^e1) + (3.0^e2) + (3.0^e3)
```

However, in geometric algebra we don't only care about the vectors, we want to transform the the higher-order blades too. This can be done via an outermorphism, which extends  $f(a)$  to  $f(a \wedge b) = f(a) \wedge f(b)$ . This is where the `clifford.transformations` submodule comes in handy:

```
[7]: from clifford import transformations

rot_and_scale_x_ga = transformations.OutermorphismMatrix(rot_and_scale_x, layout)
```

To apply these transformations, we use the `()` operator, rather than `@`:

```
[8]: rot_and_scale_x_ga(e12)

[8]: (1^e12) + (1^e13)
```

```
[9]: # check it's an outermorphism
rot_and_scale_x_ga(e1) ^ rot_and_scale_x_ga(e2)

[9]: (1^e12) + (1^e13)
```

It shouldn't come as a surprise that applying the transformation to the psuedoscalar will tell us the determinant of our original matrix - the determinant tells us how a transformation scales volumes, and `layout.I` is a representation of the unit volume element!

```
[10]: np.linalg.det(rot_and_scale_x), rot_and_scale_x_ga(layout.I)

[10]: (2.0, (2^e123))
```

### 13.2.1 Matrix representation

Under the hood, clifford implements this using a matrix too - it's just now a matrix operating over all of the basis blades, not just over the vectors. We can see this by looking at the *private* `_matrix` attribute:

```
[11]: show_table(rot_and_scale_x_ga._matrix, ["$\mathit{in}_{\{s\}}$" % c for c in layout.
↳ names], ["$\mathit{out}_{\{s\}}$" % c for c in layout.names])
```

```
[11]:      $\mathit{in}_{\{ \}}$   $\mathit{in}_{\{e1\}}$   \
$\mathit{out}_{\{ \}}$           1           0
$\mathit{out}_{\{e1\}}$         0           1
$\mathit{out}_{\{e2\}}$         0           0
$\mathit{out}_{\{e3\}}$         0           0
$\mathit{out}_{\{e12\}}$        0           0
$\mathit{out}_{\{e13\}}$        0           0
$\mathit{out}_{\{e23\}}$        0           0
$\mathit{out}_{\{e123\}}$       0           0

      $\mathit{in}_{\{e2\}}$   $\mathit{in}_{\{e3\}}$   \
$\mathit{out}_{\{ \}}$           0           0
$\mathit{out}_{\{e1\}}$         0           0
$\mathit{out}_{\{e2\}}$         1          -1
$\mathit{out}_{\{e3\}}$         1           1
$\mathit{out}_{\{e12\}}$        0           0
$\mathit{out}_{\{e13\}}$        0           0
$\mathit{out}_{\{e23\}}$        0           0
$\mathit{out}_{\{e123\}}$       0           0

      $\mathit{in}_{\{e12\}}$   $\mathit{in}_{\{e13\}}$   \
$\mathit{out}_{\{ \}}$           0           0
$\mathit{out}_{\{e1\}}$         0           0
$\mathit{out}_{\{e2\}}$         0           0
$\mathit{out}_{\{e3\}}$         0           0
$\mathit{out}_{\{e12\}}$        1          -1
$\mathit{out}_{\{e13\}}$        1           1
$\mathit{out}_{\{e23\}}$        0           0
$\mathit{out}_{\{e123\}}$       0           0

      $\mathit{in}_{\{e23\}}$   $\mathit{in}_{\{e123\}}$
$\mathit{out}_{\{ \}}$           0           0
$\mathit{out}_{\{e1\}}$         0           0
$\mathit{out}_{\{e2\}}$         0           0
$\mathit{out}_{\{e3\}}$         0           0
$\mathit{out}_{\{e12\}}$        0           0
$\mathit{out}_{\{e13\}}$        0           0
$\mathit{out}_{\{e23\}}$        2           0
$\mathit{out}_{\{e123\}}$       0           2
```

..... doc/tutorials/linear-transformations.ipynb ends here.

The following section was generated from docs/tutorials/MatrixRepresentationsOfGeometricFunctions.ipynb .....



## MATRIX REPRESENTATIONS OF GEOMETRIC FUNCTIONS

More info can be found in Chapter 5 of “New foundations for classical mechanics”, by David Hestenes.

This notebook shows how some matrix groups can be represented in geometric algebra. Not as spinors in CGA, or PGA, just as functions in plain old Geometric Algebra.

This is done by:

1. Creating a geometric function
2. Apply it to an orthonormal frame
3. Convert the resultant frame into a matrix

The matrix is defined as the inner product of each basis element of original and transformed frame.

$$M_{ij} = a_i \cdot b_j = a_i \cdot f(a)_j$$

(or vice-versa with the i,j, you get the point). Since we are going to do this repeatedly, define a `func2Mat ()`

```
[1]: import numpy as np

def func2Mat(f,I):
    '''
    Convert a function acting on a vector into a matrix, given
    the space defined by psuedoscalar I
    '''
    A = I.basis()
    B = [f(a) for a in A]
    M = [float(b | a) for a in A for b in B]
    return np.array(M).reshape(len(B), len(B))
```

Start with initializing a euclidean N-dimensional algebra and assign our pseudoscalar to *I*, pretty standard.

```
[2]: from clifford import Cl
from math import *

l,b = Cl(3)          # returns (layout,blades). you can change dimesion here
I = l.pseudoScalar
```

## 14.1 Anti-symmetric

This is so easy,

$$x \rightarrow x \cdot B$$

```
[3]: B = l.randomIntMV()(2) # we use randIntMV because its easier to read
f = lambda x:x|B
func2Mat(f,I=I)

[3]: array([[ 0., -3.,  9.],
           [ 3.,  0., -6.],
           [-9.,  6.,  0.]])
```

Whats the B? you can read its values straight off the matrix.

```
[4]: B
[4]: (3^e12) - (9^e13) + (6^e23)
```

## 14.2 Diagonal ( Directional Scaling)

A bit awkward this one, but its made by projection onto each basis vector, then scaling the component by some amount.

$$x \rightarrow \sum \lambda_i (x \cdot e_i) e_i$$

```
[5]: ls = range(1,len(I.basis())+1) # some dilation values (eigenvalues)
A = I.basis()

d = lambda x: sum([(x|a)/a*1 for a,l in zip(A,ls)])
func2Mat(d,I=I)

[5]: array([[1., 0., 0.],
           [0., 2., 0.],
           [0., 0., 3.]])
```

## 14.3 Orthogonal, Rotation

$$x \rightarrow Rx\tilde{R}$$

where

$$R = e^{B/2}$$



```
[6]: B = l.randomMV() (2)
      R = e**(B/2)
      r = lambda x: R*x*~R
      func2Mat(r,I=I)

[6]: array([[ -0.02877844,  0.83278728,  0.5528446 ],
            [ -0.2119728 , -0.54557927,  0.81080873],
            [ 0.97685175, -0.09385421,  0.19222917]])
```

The inverse of this is ,

$$x \rightarrow \tilde{R}xR$$

```
[7]: rin = lambda x: ~R*x*R # the inverse rotation
      func2Mat(rin,I=I)

[7]: array([[ -0.02877844, -0.2119728 ,  0.97685175],
            [ 0.83278728, -0.54557927, -0.09385421],
            [ 0.5528446 ,  0.81080873,  0.19222917]])
```

## 14.4 Orthogonal, Reflection

$$x \rightarrow -axa^{-1}$$

```
[8]: a = l.randomIntMV() (1)
      n = lambda x: -a*x/a
      func2Mat(n,I=I)

[8]: array([[ 0.97530864, -0.09876543, -0.19753086],
            [-0.09876543,  0.60493827, -0.79012346],
            [-0.19753086, -0.79012346, -0.58024691]])
```

```
[9]: a

[9]: -(1^e1) - (4^e2) - (8^e3)
```

Notice the determinant for reflection is -1, and for rotation is +1.

```
[10]: from numpy.linalg import det
       det(func2Mat(n,I=I)), det(func2Mat(r,I=I))

[10]: (-0.9999999999999999, 1.0000000000411782)
```

## 14.5 Symmetric

This can be built up from the functions we just defined ie  $\text{Rotation} * \text{Dilation} / \text{Rotation}$

$$x \rightarrow r(d(r^{-1}(x)))$$

which if you write it out, looks kind of dumb

$$x \rightarrow R[\sum \lambda_i(\tilde{R}xR \cdot e_i)e_i]R$$

So, the antisymmetric matrix is interpreted as a set dilations about a some orthogonal frame rotated from the basis (what basis, eh? exactly what basis!).

More generally we could include reflection in the  $R$  too.

```
[11]: g = lambda x: r(d(rinv(x)))
      func2Mat(g, I=I)
[11]: array([[2.30480895, 0.44215098, 0.13438513],
            [0.44215098, 2.61247833, 0.36292709],
            [0.13438513, 0.36292709, 1.08271272]])
```

## 14.6 Eigen stuffs

By definition the eigen-stuff is the invariants of the transformation, sometimes this is a vector, and other times it is a plane.

### 14.6.1 Rotation

The eigen blades of a rotation are really the axis and plane of rotation.

```
[12]: from numpy.linalg import eig
      vals, vecs = eig(func2Mat(r, I=I))
      np.round(vecs, 3)
[12]: array([[ 0.626+0.j, -0.136-0.535j, -0.136+0.535j],
            [ 0.293+0.j,  0.676+0.j,  0.676-0.j ],
            [ 0.723+0.j, -0.157+0.463j, -0.157-0.463j]])
```

If you checkout the real column, and compare this to the bivector which generated this rotation (aka the generator), after its been normalized

```
[13]: B/(abs(B))
[13]: (0.72272^e12) - (0.29331^e13) + (0.62581^e23)
[14]: B
[14]: (1.68666^e12) - (0.68452^e13) + (1.46049^e23)
[15]: vals
```

```
[15]: array([ 1.          +0.j          , -0.69106427+0.72279332j,
          -0.69106427-0.72279332j])
```

```
[16]: cos(abs(B)), sin(abs(B))
```

```
[16]: (-0.6910642689158123, 0.7227933150132574)
```

## 14.6.2 Symmetric

For the symmetric matrix, the invariant thing the orthonormal frame along which the dilations take place

```
[17]: vals, vecs = eig(func2Mat(g,I=I))
      np.round(vecs,5).T
```

```
[17]: array([[ 0.55284,  0.81081,  0.19223],
            [ 0.83279, -0.54558, -0.09385],
            [-0.02878, -0.21197,  0.97685]])
```

This is easily found by using the rotation part of the symmetric operator,

```
[18]: [R*a~R for a in I.basis()]
```

```
[18]: [-(0.02878^e1) - (0.21197^e2) + (0.97685^e3),
      (0.83279^e1) - (0.54558^e2) - (0.09385^e3),
      (0.55284^e1) + (0.81081^e2) + (0.19223^e3)]
```

## 14.7 Primitive Visualization in 2D

```
[19]: from pylab import linspace, plot,axis,legend
```

```
def plot_ps(ps,**kw):
    x = [p[e1] for p in ps]
    y = [p[e2] for p in ps]
    plot(x,y, marker='o',ls=' ',**kw)

l,b = Cl(2)
locals().update(b)
I = l.pseudoScalar

## define function of interest
B = l.randomMV()(2)
R = e**(B/2)
f = lambda x: R*x~R

## loop though cartesian grid and apply f,
ps,qs=[],[]
for x in linspace(-1,1,11):
    for y in linspace(-1,1,11):
        p = x*e1 + y*e2
        q = f(p)
        ps.append(p)
        qs.append(q)
```

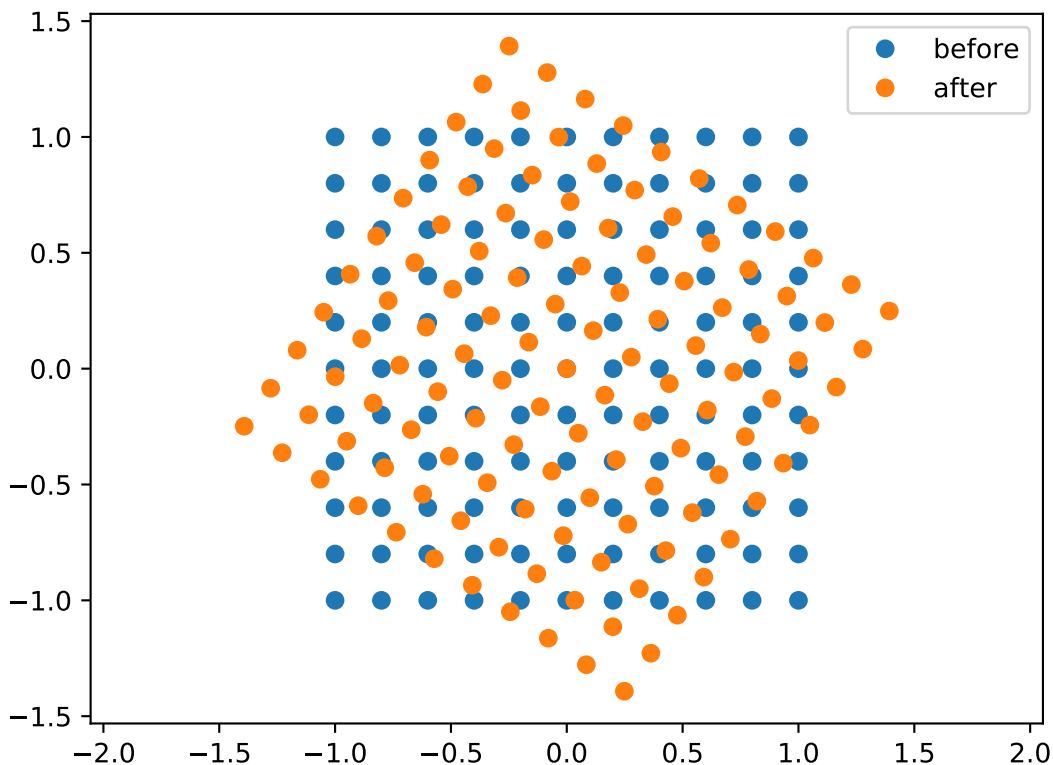
(continues on next page)

(continued from previous page)

```
plot_ps(ps, label='before')
plot_ps(qs, label='after')
axis('equal')
legend()
```

```
/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
python3.8/site-packages/traitlets/traitlets.py:3030: FutureWarning: --rc={'figure.
dpi': 96, 'savefig.transparent': True} for dict-traits is deprecated in traitlets 5.
0. You can pass --rc <key=value> ... multiple times to add items to a dict.
warn(
```

```
[19]: <matplotlib.legend.Legend at 0x7f6a014c8a30>
```



```
[20]: func2Mat(f, I = I )
```

```
[20]: array([[ 0.82037719,  0.57182276],
          [-0.57182276,  0.82037719]])
```

..... doc/tutorials/MatrixRepresentationsOfGeometricFunctions.ipynb ends here.

The following section was generated from docs/tutorials/apollonius-cga-augmented.ipynb .....

## WORKING WITH CUSTOM ALGEBRAS

This notebook explores the algebra defined in [The Lie Model for Euclidean Geometry \(Hongbo Li\)](#), and its application to solving [Apollonius' Problem](#). It also shows

The algebra is constructed with basis elements  $e_{-2}, e_{-1}, e_1, \dots, e_n, e_{n+1}$ , where  $e_{-2}^2 = -e_{-1}^2 = -e_{n+1}^2 = 1$ . This is an extension of a standard conformal algebra, with an extra  $e_{n+1}$  basis vector.

Note that we permuted the order in the source code below to make `ConformalLayout` happy.

```
[1]: from clifford import ConformalLayout, BasisVectorIds, MultiVector, transformations

class OurCustomLayout(ConformalLayout):
    def __init__(self, ndims):
        self.ndims = ndims

        euclidean_vectors = [str(i + 1) for i in range(ndims)]
        conformal_vectors = ['m2', 'm1']

        # Construct our custom algebra. Note that ConformalLayout requires the e- and
        # e+ basis vectors to be last.
        ConformalLayout.__init__(
            self,
            [1]*ndims + [-1] + [1, -1],
            ids=BasisVectorIds(euclidean_vectors + ['np1'] + conformal_vectors)
        )
        self.enp1 = self.basis_vectors_lst[ndims]

        # Construct a base algebra without the extra `enp1`, which would not be
        # understood by pyganja.
        self.conformal_base = ConformalLayout(
            [1]*ndims + [1, -1],
            ids=BasisVectorIds(euclidean_vectors + conformal_vectors)
        )

        # this lets us convert between the two layouts
        self.to_conformal = transformations.between_basis_vectors(self, self.
        conformal_base)
```

The code above also defines a standard conformal  $\mathbb{R}^{N+1,1}$  layout without this new basis vector. This is primarily to support rendering with `pyganja`, which doesn't support the presence of this extra vector. `BasisVectorMap` defaults to preserving vectors by name between one algebra and another, while throwing away blades containing vectors missing from the destination algebra.

We define an `ups` function which maps conformal dual-spheres into this algebra, as  $s' = s + |s|e_{n+1}$ , and a `downs` that applies the correct sign. The `s` suffix here is chosen to mean `sphere`.

```
[2]: def ups(self, s):
      return s + self.enp1*abs(s)

OurCustomLayout.ups = ups; del ups

def downs(self, mv):
    if (mv | self.enp1) [()] > 0:
        mv = -mv
    return mv

OurCustomLayout.downs = downs; del downs
```

Before we start looking at specified dimensions of euclidean space, we build a helper to construct conformal dual circles and spheres, with the word `round` being a general term intended to cover both circles and spheres.

```
[3]: def dual_round(at, r):
      l = at.layout
      return l.up(at) - 0.5*l.einf*r*r
```

In order to render with `pyganja`, we'll need a helper to convert from our custom  $\mathbb{R}^{N+1,2}$  layout into a standard conformal  $\mathbb{R}^{N+1,1}$  layout. `clifford` maps indices in `.value` to basis blades via `layout._basis_blade_order.index_to_bitmap`, which we can use to convert the indices in one layout to the indices in another.

## 15.1 Visualization

Finally, we'll define a plotting function, which plots the problem and solution circles in suitable colors via `pyganja`. Note that this all works because of our definition of the `to_conformal BasisVectorMap`.

```
[4]: import itertools
      from pyganja import GanjaScene, draw

def plot_rounds(in_rounds, out_rounds, scale=1):
    colors = itertools.cycle([
        (255, 0, 0),
        (0, 255, 0),
        (0, 0, 255),
        (0, 255, 255),
    ])
    # note: .dual() needed here because we're passing in dual rounds, but ganja_
    ↳ expects direct rounds
    s = GanjaScene()
    for r, color in zip(in_rounds, colors):
        s.add_object(r.layout.to_conformal(r).dual(), color=color)
    for r in out_rounds:
        s.add_object(r.layout.to_conformal(r).dual(), color=(64, 64, 64))
    draw(s, sig=r.layout.conformal_base.sig, scale=scale)

/home/docs/checkouts/readthedocs.org/user_builds/clifford-alex/envs/latest/lib/
↳ python3.8/site-packages/pyganja/__init__.py:2: UserWarning: Failed to import cef_
↳ gui, cef functions will be unavailable
    from .script_api import *
```

## 15.2 Apollonius' problem in $\mathbb{R}^2$ with circles

```
[5]: l2 = OurCustomLayout(ndims=2)
     e1, e2 = l2.basis_vectors_lst[:2]
```

This gives us the Layout l2 with the desired metric,

```
[6]: import pandas as pd # convenient but somewhat slow trick for showing tables
     pd.DataFrame(l2.metric, index=l2.basis_names, columns=l2.basis_names)

[6]:      e1    e2  enp1  em2  em1
e1      1.0  0.0   0.0  0.0  0.0
e2      0.0  1.0   0.0  0.0  0.0
enp1    0.0  0.0  -1.0  0.0  0.0
em2     0.0  0.0   0.0  1.0  0.0
em1     0.0  0.0   0.0  0.0 -1.0
```

Now we can build some dual circles:

```
[7]: # add minus signs before `dual_round` to flip circle directions
     c1 = dual_round(-e1-e2, 1)
     c2 = dual_round(e1-e2, 0.75)
     c3 = dual_round(e2, 0.5)
```

Compute the space orthogonal to all of them, which is an object of grade 2:

```
[8]: pp = (l2.ups(c1) ^ l2.ups(c2) ^ l2.ups(c3)).dual()
     pp.grades()

[8]: {2}
```

We hypothesize that this object is of the form  $l2.ups(c4) \wedge l2.ups(c5)$ . Taking a step not mentioned in the original paper, we decide to treat this as a regular conformal point pair, which allows us to project out the two factors with the approach taken in [LLW04]. Here, we normalize with  $e_{n+1}$  instead of the usual  $n_\infty$ :

```
[9]: def pp_ends(pp):
     P = (1 + pp.normal()) / 2
     return P * (pp | pp.layout.enp1), ~P * (pp | pp.layout.enp1)

     c4u, c5u = pp_ends(pp)
```

And finally, plot our circles:

```
[10]: plot_rounds([c1, c2, c3], [l2.downs(c4u), l2.downs(c5u)], scale=0.75)

<IPython.core.display.Javascript object>
```

This works for colinear circles too:

```
[11]: c1 = dual_round(-1.5*e1, 0.5)
     c2 = dual_round(e1*0, 0.5)
     c3 = dual_round(1.5*e1, 0.5)
     c4u, c5u = pp_ends((l2.ups(c1) ^ l2.ups(c2) ^ l2.ups(c3)).dual())

     plot_rounds([c1, c2, c3], [l2.downs(c4u), l2.downs(c5u)])

<IPython.core.display.Javascript object>
```

```
[12]: c1 = dual_round(-3*e1, 1.5)
      c2 = dual_round(-2*e1, 1)
      c3 = -dual_round(2*e1, 1)
      c4u, c5u = pp_ends((l2.ups(c1) ^ l2.ups(c2) ^ l2.ups(c3)).dual())

      plot_rounds([c1, c2, c3], [l2.downs(c4u), l2.downs(c5u)])

      <IPython.core.display.Javascript object>
```

## 15.3 Apollonius' problem in $\mathbb{R}^3$ with spheres

```
[13]: l3 = OurCustomLayout(ndims=3)
      e1, e2, e3 = l3.basis_vectors_lst[:3]
```

Again, we can check the metric:

```
[14]: pd.DataFrame(l3.metric, index=l3.basis_names, columns=l3.basis_names)

[14]:
```

	e1	e2	e3	enp1	em2	em1
e1	1.0	0.0	0.0	0.0	0.0	0.0
e2	0.0	1.0	0.0	0.0	0.0	0.0
e3	0.0	0.0	1.0	0.0	0.0	0.0
enp1	0.0	0.0	0.0	-1.0	0.0	0.0
em2	0.0	0.0	0.0	0.0	1.0	0.0
em1	0.0	0.0	0.0	0.0	0.0	-1.0

And apply the solution to some spheres, noting that we now need 4 in order to constrain our solution

```
[15]: c1 = dual_round(e1+e2+e3, 1)
      c2 = dual_round(-e1+e2-e3, 0.25)
      c3 = dual_round(e1-e2-e3, 0.5)
      c4 = dual_round(-e1-e2+e3, 1)
      c5u, c6u = pp_ends((l3.ups(c1) ^ l3.ups(c2) ^ l3.ups(c3) ^ l3.ups(c4)).dual())

      plot_rounds([c1, c2, c3, c4], [l3.downs(c6u), l3.downs(c5u)], scale=0.25)

      <IPython.core.display.Javascript object>
```

Note that the figure above can be rotated!

..... doc/tutorials/apollonius-cga-augmented.ipynb ends here.



## OTHER RESOURCES FOR CLIFFORD

### 16.1 Slide Decks

- [Installation](#)
- [Conformal Geometric Algebra with Python, Part 1](#)
- [Conformal Geometric Algebra with Python, Part 2](#)

### 16.2 Videos

- [Intro to Clifford](#)



## OTHER RESOURCES FOR GEOMETRIC ALGEBRA

If you think Geometric Algebra looks interesting and want to learn more, check out these websites and textbooks

### 17.1 Links

- [galgebra](#), a symbolic geometric algebra module for Python
- [The Cambridge University Geometric Algebra Research Group home page](#)
- [David Hestenes' Geometric Calculus R & D Home Page](#)

### 17.2 Introductory textbooks

- [Geometric Algebra for Physicists](#), by Doran and Lasenby
- [Geometric Algebra for Computer Science](#), by Dorst, Fontijne and Mann
- [New Foundations for Classical Mechanics](#), by David Hestenes



**BIBLIOGRAPHY**



## BIBLIOGRAPHY

- [Cor17] Allan Cortzen. Geometric algebra algorithm representing an orthogonal isomorphism in versor form v2.03. Available online, 10 2017. URL: <https://ctz.dk/geometric-algebra/frames-to-versor-algorithm/>.
- [DL03] Chris Doran and Anthony Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003. URL: <http://www.mrao.cam.ac.uk/~clifford>.
- [DFM09] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry*. Morgan Kaufmann Publishers Inc., 2009. ISBN 9780080553108.
- [DV11] Leo Dorst and Robert Valkenburg. *Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition*, pages 81–104. Springer London, 01 2011. doi:10.1007/978-0-85729-811-9\_5.
- [FD11] Daniel Fontijne and Leo Dorst. Reconstructing rotations and rigid body motions from exact point correspondences through reflections. *Journal of Theoretical Biology - J THEOR BIOL*, pages 63–78, 01 2011. doi:10.1007/978-0-85729-811-9\_4.
- [HL19] Hugo Hadfield and Joan Lasenby. Direct linear interpolation of geometric objects in conformal geometric algebra. *Advances in Applied Clifford Algebras*, 29:, 09 2019. doi:10.1007/s00006-019-1003-y.
- [Hes15] David Hestenes. *Space-time algebra*. Springer Berlin Heidelberg, New York, second edition edition, 2015. ISBN 978-3-319-18412-8 978-3-319-18413-5.
- [HS17] Eckhard Hitzer and Stephen Sangwine. Multivector and multivector matrix inverses in real clifford algebras. *Applied Mathematics and Computation*, 311:375–389, Oct 2017. doi:10.1016/j.amc.2017.05.027.
- [LLW04] Anthony Lasenby, Joan Lasenby, and Rich Wareham. A Covariant Approach to Geometry using Geometric Algebra. Technical Report F-INFENG/TR-483, Department of Engineering, University of Cambridge, 2004. URL: <https://pdfs.semanticscholar.org/baba/976fd7f6577eeaa1d3ef488c1db13ec24652.pdf>.
- [LHL19] Joan Lasenby, Hugo Hadfield, and Anthony Lasenby. Calculating the rotor between conformal objects. *Advances in Applied Clifford Algebras*, 29:102, 10 2019. doi:10.1007/s00006-019-1014-8.
- [WCL05] Rich Wareham, Jonathan Cameron, and Joan Lasenby. Applications of conformal geometric algebra in computer vision and graphics. In Hongbo Li, Peter J. Olver, and Gerald Sommer, editors, *Computer Algebra and Geometric Algebra with Applications*, 329–349. Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.
- [WL08] Rich Wareham and Joan Lasenby. Mesh vertex pose and position interpolation using geometric algebra. In *Articulated Motion and Deformable Objects*, 122–131. Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.





## PYTHON MODULE INDEX

### C

- `clifford`, 5
- `clifford.cga`, 18
- `clifford.dg3c`, 69
- `clifford.dpga`, 69
- `clifford.g2`, 69
- `clifford.g2c`, 69
- `clifford.g3`, 69
- `clifford.g3c`, 69
- `clifford.g4`, 69
- `clifford.gac`, 69
- `clifford.operator`, 63
- `clifford.pga`, 69
- `clifford.tools`, 30
- `clifford.tools.classify`, 57
- `clifford.tools.g3`, 31
- `clifford.tools.g3c`, 34
- `clifford.tools.g3c.object_fitting`, 55
- `clifford.transformations`, 63



## Symbols

`__add__()` (*clifford.MultiVector method*), 7  
`__and__()` (*clifford.MultiVector method*), 7  
`__call__()` (*clifford.MultiVector method*), 8  
`__getitem__()` (*clifford.MultiVector method*), 8  
`__init__()` (*clifford.cga.CGA method*), 20  
`__init__()` (*clifford.cga.CGAThing method*), 30  
`__init__()` (*clifford.cga.Dilation method*), 26  
`__init__()` (*clifford.cga.Flat method*), 22  
`__init__()` (*clifford.cga.Rotation method*), 25  
`__init__()` (*clifford.cga.Round method*), 24  
`__init__()` (*clifford.cga.Translation method*), 28  
`__init__()` (*clifford.cga.Transversion method*), 29  
`__invert__()` (*clifford.MultiVector method*), 8  
`__mul__()` (*clifford.MultiVector method*), 7  
`__or__()` (*clifford.MultiVector method*), 7  
`__sub__()` (*clifford.MultiVector method*), 7  
`__xor__()` (*clifford.MultiVector method*), 7

## A

`adjoint()` (*clifford.MultiVector method*), 8  
`adjoint()` (*clifford.transformations.LinearMatrix property*), 64  
`adjoint_func` (*clifford.Layout attribute*), 13  
`angle_between_vectors()` (*in module clifford.tools.g3c*), 32  
`annihilate_k()` (*in module clifford.tools.g3c*), 52  
`anticommutator()` (*clifford.MultiVector method*), 10  
`apply_rotor()` (*in module clifford.tools.g3c*), 45  
`apply_rotor_inv()` (*in module clifford.tools.g3c*), 46  
`as_array()` (*clifford.MultiVector method*), 8  
`astype()` (*clifford.MultiVector method*), 11  
`average_objects()` (*in module clifford.tools.g3c*), 53

## B

`b1()` (*clifford.BladeMap property*), 17  
`b2()` (*clifford.BladeMap property*), 17  
`base_vector()` (*clifford.cga.CGA method*), 20  
`bases()` (*clifford.Layout method*), 14  
`basis()` (*clifford.MultiVector method*), 10

`basis_names()` (*clifford.Layout property*), 14  
`basis_vectors()` (*clifford.Layout property*), 14  
`basis_vectors_lst()` (*clifford.Layout property*), 14  
`between_basis_vectors()` (*in module clifford.transformations*), 66  
`Blade` (*class in clifford.tools.classify*), 57  
`BladeMap` (*class in clifford*), 17  
`blades` (*in module clifford.<predefined>*), 69  
`blades()` (*clifford.Layout property*), 14  
`blades_list()` (*clifford.Layout property*), 14  
`blades_list()` (*clifford.MultiVector property*), 9  
`blades_of_grade()` (*clifford.Layout method*), 14  
`bladeTupList` (*clifford.Layout attribute*), 12

## C

`calculate_S_over_mu()` (*in module clifford.tools.g3c*), 55  
`center()` (*clifford.cga.Round property*), 23  
`center_down()` (*clifford.cga.Round property*), 23  
`CGA` (*class in clifford.cga*), 19  
`CGAThing` (*class in clifford.cga*), 29  
`check_infinite_roots()` (*in module clifford.tools.g3c*), 51  
`check_infinite_roots_val()` (*in module clifford.tools.g3c*), 51  
`check_sigma_for_negative_root()` (*in module clifford.tools.g3c*), 51  
`check_sigma_for_negative_root_val()` (*in module clifford.tools.g3c*), 51  
`check_sigma_for_positive_root()` (*in module clifford.tools.g3c*), 51  
`check_sigma_for_positive_root_val()` (*in module clifford.tools.g3c*), 51  
`Circle` (*class in clifford.tools.classify*), 59  
`circle_to_sphere()` (*in module clifford.tools.g3c*), 40  
`Cl()` (*in module clifford*), 5  
`classify()` (*in module clifford.tools.classify*), 57  
`clean()` (*clifford.MultiVector method*), 8  
`clifford`  
     *module*, 5

`clifford.cga`  
     module, 18  
`clifford.dg3c`  
     module, 69  
`clifford.dpga`  
     module, 69  
`clifford.g2`  
     module, 69  
`clifford.g2c`  
     module, 69  
`clifford.g3`  
     module, 69  
`clifford.g3c`  
     module, 69  
`clifford.g4`  
     module, 69  
`clifford.gac`  
     module, 69  
`clifford.operator`  
     module, 63  
`clifford.pga`  
     module, 69  
`clifford.tools`  
     module, 30  
`clifford.tools.classify`  
     module, 57  
`clifford.tools.g3`  
     module, 31  
`clifford.tools.g3c`  
     module, 34  
`clifford.tools.g3c.object_fitting`  
     module, 55  
`clifford.transformations`  
     module, 63  
`closest_point_on_circle_from_line()` (in  
     module `clifford.tools.g3c`), 43  
`closest_point_on_line_from_circle()` (in  
     module `clifford.tools.g3c`), 43  
`commutator()` (*clifford.MultiVector* method), 9  
`conformalize()` (in module *clifford*), 6  
`ConformalLayout` (class in *clifford*), 15  
`conjugate()` (*clifford.MultiVector* method), 10  
`convert_2D_point_to_conformal()` (in mod-  
     ule *clifford.tools.g3c*), 41  
`convert_2D_polar_line_to_conformal_line()`  
     (in module *clifford.tools.g3c*), 40  
`correlation_matrix()` (in module *clif-*  
     *ford.tools.g3*), 33

## D

`Dilation` (class in *clifford.cga*), 26  
`dilation()` (*clifford.cga.CGA* method), 20  
`dim()` (*clifford.cga.Round* property), 23  
`dims` (*clifford.Layout* attribute), 12

`Direction` (class in *clifford.tools.classify*), 58  
`direction` (*clifford.tools.classify.Direction* attribute),  
     58  
`direction` (*clifford.tools.classify.Flat* attribute), 58  
`direction` (*clifford.tools.classify.Round* attribute), 58  
`distance_polar_line_to_euc_point_2d()`  
     (in module *clifford.tools.g3c*), 41  
`disturb_object()` (in module *clifford.tools.g3c*), 47  
`dorst_norm_val()` (in module *clifford.tools.g3c*), 50  
`down()` (*clifford.ConformalLayout* method), 15  
`dual()` (*clifford.cga.Round* property), 23  
`dual()` (*clifford.MultiVector* method), 9  
`dual_func` (*clifford.Layout* attribute), 13  
`dual_func()` (in module *clifford.tools.g3c*), 47  
`DualFlat` (class in *clifford.tools.classify*), 58

## E

`E0` (*clifford.ConformalLayout* attribute), 15  
`einf` (*clifford.ConformalLayout* attribute), 15  
`en` (*clifford.ConformalLayout* attribute), 15  
`En()` (*clifford.Frame* property), 17  
`eo` (*clifford.ConformalLayout* attribute), 15  
`ep` (*clifford.ConformalLayout* attribute), 15  
`eps()` (in module *clifford*), 16  
`euc_cross_prod()` (in module *clifford.tools.g3*), 33  
`euc_dist()` (in module *clifford.tools.g3c*), 46  
`euc_mv_to_np()` (in module *clifford.tools.g3*), 33  
`even()` (*clifford.MultiVector* property), 10  
`exp()` (*clifford.MultiVector* method), 7

## F

`factorise()` (*clifford.MultiVector* method), 10  
`fast_down()` (in module *clifford.tools.g3c*), 47  
`fast_dual()` (in module *clifford.tools.g3c*), 47  
`fast_up()` (in module *clifford.tools.g3c*), 47  
`firstIdx()` (*clifford.Layout* property), 13  
`fit_circle()` (in module *clif-*  
     *ford.tools.g3c.object\_fitting*), 56  
`fit_line()` (in module *clif-*  
     *ford.tools.g3c.object\_fitting*), 56  
`fit_plane()` (in module *clif-*  
     *ford.tools.g3c.object\_fitting*), 56  
`fit_sphere()` (in module *clif-*  
     *ford.tools.g3c.object\_fitting*), 56  
`FixedLayout` (class in *clifford.transformations*), 64  
`Flat` (class in *clifford.cga*), 21  
`Flat` (class in *clifford.tools.classify*), 58  
`flat` (*clifford.tools.classify.DualFlat* attribute), 58  
`flat()` (*clifford.cga.CGA* method), 20  
`Frame` (class in *clifford*), 17  
`frame2Mat()` (in module *clifford.tools*), 61  
`from_center_radius()` (*clifford.cga.Round*  
     method), 24

`from_function()` (*clifford.transformations.LinearMatrix class method*), 64  
`from_rotor()` (*clifford.transformations.LinearMatrix class method*), 65  
`func2Mat()` (*in module clifford.tools*), 62

## G

`GA_SVD()` (*in module clifford.tools.g3c*), 33  
`gaDims` (*clifford.Layout attribute*), 12  
`general_object_interpolation()` (*in module clifford.tools.g3c*), 53  
`general_root()` (*in module clifford.tools.g3c*), 52  
`general_root_val()` (*in module clifford.tools.g3c*), 52  
`generate_dilation_rotor()` (*in module clifford.tools.g3c*), 37  
`generate_n_clusters()` (*in module clifford.tools.g3c*), 36  
`generate_random_object_cluster()` (*in module clifford.tools.g3c*), 36  
`generate_rotation_rotor()` (*in module clifford.tools.g3c*), 32  
`generate_translation_rotor()` (*in module clifford.tools.g3c*), 37  
`get_center_from_sphere()` (*in module clifford.tools.g3c*), 39  
`get_circle_in_euc()` (*in module clifford.tools.g3c*), 39  
`get_grade_projection_matrix()` (*clifford.Layout method*), 13  
`get_left_gmt_matrix()` (*clifford.Layout method*), 13  
`get_line_intersection()` (*in module clifford.tools.g3c*), 42  
`get_line_reflection_matrix()` (*in module clifford.tools.g3c*), 48  
`get_nearest_plane_point()` (*in module clifford.tools.g3c*), 40  
`get_plane_normal()` (*in module clifford.tools.g3c*), 40  
`get_plane_origin_distance()` (*in module clifford.tools.g3c*), 40  
`get_radius_from_sphere()` (*in module clifford.tools.g3c*), 39  
`get_right_gmt_matrix()` (*clifford.Layout method*), 13  
`gmt` (*clifford.Layout attribute*), 12  
`gmt_func` (*clifford.Layout attribute*), 13  
`gmt_func_generator()` (*clifford.Layout method*), 13  
`gp()` (*in module clifford.operator*), 63  
`grade_mask()` (*clifford.Layout method*), 13  
`grade_obj()` (*in module clifford*), 18

`gradeInvol()` (*clifford.MultiVector method*), 10  
`gradeList()` (*clifford.Layout property*), 12  
`grades()` (*clifford.MultiVector method*), 9

## H

`hitzer_inverse()` (*clifford.MultiVector method*), 9  
`homo()` (*clifford.ConformalLayout method*), 15

## I

`I()` (*clifford.Layout property*), 14  
`I()` (*clifford.MultiVector property*), 9  
`I_base` (*clifford.ConformalLayout attribute*), 15  
`imt` (*clifford.Layout attribute*), 12  
`imt_func` (*clifford.Layout attribute*), 13  
`imt_func_generator()` (*clifford.Layout method*), 13  
`InfinitePoint` (*class in clifford.tools.classify*), 59  
`interp_objects_root()` (*in module clifford.tools.g3c*), 53  
`interpret_multivector_as_object()` (*in module clifford.tools.g3c*), 48  
`intersect_line_and_plane_to_point()` (*in module clifford.tools.g3c*), 39  
`inv()` (*clifford.Frame property*), 17  
`inv()` (*clifford.MultiVector method*), 9  
`inv_func` (*clifford.Layout attribute*), 13  
`inverted()` (*clifford.cga.CGAThing method*), 30  
`inverted()` (*clifford.cga.Dilation method*), 27  
`inverted()` (*clifford.cga.Flat method*), 22  
`inverted()` (*clifford.cga.Rotation method*), 26  
`inverted()` (*clifford.cga.Round method*), 24  
`inverted()` (*clifford.cga.Translation method*), 28  
`inverted()` (*clifford.cga.Transversion method*), 29  
`involuted()` (*clifford.cga.CGAThing method*), 30  
`involuted()` (*clifford.cga.Dilation method*), 27  
`involuted()` (*clifford.cga.Flat method*), 22  
`involuted()` (*clifford.cga.Rotation method*), 26  
`involuted()` (*clifford.cga.Round method*), 24  
`involuted()` (*clifford.cga.Translation method*), 28  
`involuted()` (*clifford.cga.Transversion method*), 29  
`invPS()` (*clifford.MultiVector method*), 9  
`ip()` (*in module clifford.operator*), 63  
`is_innerymorphic_to()` (*clifford.Frame method*), 17  
`isBlade()` (*clifford.MultiVector method*), 9  
`isScalar()` (*clifford.MultiVector method*), 9  
`isVersor()` (*clifford.MultiVector method*), 9  
`iterative_closest_points_circle_line()` (*in module clifford.tools.g3c*), 43  
`iterative_closest_points_on_circles()` (*in module clifford.tools.g3c*), 42  
`iterative_furthest_points_on_circles()` (*in module clifford.tools.g3c*), 43

## J

`join()` (*clifford.MultiVector method*), 10

## L

`Layout` (class in *clifford*), 11

`layout` (*clifford.tools.classify.Blade attribute*), 58

`layout` (in module *clifford*.<predefined>), 69

`layout1()` (*clifford.BladeMap property*), 17

`layout2()` (*clifford.BladeMap property*), 17

`lc()` (*clifford.MultiVector method*), 8

`lcmt` (*clifford.Layout attribute*), 12

`lcmt_func` (*clifford.Layout attribute*), 13

`lcmt_func_generator()` (*clifford.Layout method*), 13

`left_complement()` (*clifford.MultiVector method*), 8

`left_complement_func` (*clifford.Layout attribute*), 13

`leftInv()` (*clifford.MultiVector method*), 9

`leftLaInv()` (*clifford.MultiVector method*), 9

`Line` (class in *clifford.tools.classify*), 59

`line_to_point_and_direction()` (in module *clifford.tools.g3c*), 40

`Linear` (class in *clifford.transformations*), 64

`LinearMatrix` (class in *clifford.transformations*), 64

`load_ga_file()` (*clifford.Layout method*), 13

`location` (*clifford.tools.classify.Flat attribute*), 58

`location` (*clifford.tools.classify.Round attribute*), 58

## M

`mag2()` (*clifford.MultiVector method*), 8

`mat2Frame()` (in module *clifford.tools*), 61

`meet()` (*clifford.MultiVector method*), 11

`meet()` (in module *clifford.tools.g3c*), 45

`meet_val()` (in module *clifford.tools.g3c*), 45

`metric()` (*clifford.Layout property*), 13

`midpoint_between_lines()` (in module *clifford.tools.g3c*), 41

`midpoint_of_line_cluster()` (in module *clifford.tools.g3c*), 41

module

`clifford`, 5

`clifford.cga`, 18

`clifford.dg3c`, 69

`clifford.dpga`, 69

`clifford.g2`, 69

`clifford.g2c`, 69

`clifford.g3`, 69

`clifford.g3c`, 69

`clifford.g4`, 69

`clifford.gac`, 69

`clifford.operator`, 63

`clifford.pga`, 69

`clifford.tools`, 30

`clifford.tools.classify`, 57

`clifford.tools.g3`, 31

`clifford.tools.g3c`, 34

`clifford.tools.g3c.object_fitting`, 55

`clifford.transformations`, 63

`mult_with_ninf()` (in module *clifford.tools.g3c*), 46

`MultiVector` (class in *clifford*), 7

`MultiVector()` (*clifford.Layout method*), 14

`mv()` (*clifford.tools.classify.Blade property*), 58

## N

`n_th_rotor_root()` (in module *clifford.tools.g3c*), 53

`names` (*clifford.Layout attribute*), 12

`neg_twiddle_root()` (in module *clifford.tools.g3c*), 53

`neg_twiddle_root_val()` (in module *clifford.tools.g3c*), 52

`negative_root()` (in module *clifford.tools.g3c*), 52

`negative_root_val()` (in module *clifford.tools.g3c*), 51

`norm()` (in module *clifford.tools.g3c*), 46

`normal()` (*clifford.MultiVector method*), 9

`normalInv()` (*clifford.MultiVector method*), 9

`normalise_n_minus_1()` (in module *clifford.tools.g3c*), 45

`normalise_TR_to_unit_T()` (in module *clifford.tools.g3c*), 48

`normalised()` (in module *clifford.tools.g3c*), 46

`np_to_euc_mv()` (in module *clifford.tools.g3*), 33

`null_vector()` (*clifford.cga.CGA method*), 20

## O

`odd()` (*clifford.MultiVector property*), 10

`omoh()` (in module *clifford.tools*), 62

`omt` (*clifford.Layout attribute*), 12

`omt_func` (*clifford.Layout attribute*), 13

`omt_func_generator()` (*clifford.Layout method*), 13

`op()` (in module *clifford.operator*), 63

`orthoFrames2Versor()` (in module *clifford.tools*), 60

`orthoMat2Versor()` (in module *clifford.tools*), 61

`OutermorphismMatrix` (class in *clifford.transformations*), 65

## P

`parse_multivector()` (*clifford.Layout method*), 13

`Plane` (class in *clifford.tools.classify*), 59

`Point` (class in *clifford.tools.classify*), 59

`point_pair_to_end_points()` (in module *clifford.tools.g3c*), 39



[PointFlat \(class in clifford.tools.classify\)](#), 59  
[PointPair \(class in clifford.tools.classify\)](#), 59  
[pos\\_twiddle\\_root\(\)](#) (in module [clifford.tools.g3c](#)), 52  
[pos\\_twiddle\\_root\\_val\(\)](#) (in module [clifford.tools.g3c](#)), 52  
[positive\\_root\(\)](#) (in module [clifford.tools.g3c](#)), 51  
[positive\\_root\\_val\(\)](#) (in module [clifford.tools.g3c](#)), 51  
[pretty\(\)](#) (in module [clifford](#)), 16  
[print\\_precision\(\)](#) (in module [clifford](#)), 16  
[project\(\)](#) ([clifford.MultiVector](#) method), 10  
[project\\_points\\_to\\_circle\(\)](#) (in module [clifford.tools.g3c](#)), 42  
[project\\_points\\_to\\_line\(\)](#) (in module [clifford.tools.g3c](#)), 42  
[project\\_points\\_to\\_plane\(\)](#) (in module [clifford.tools.g3c](#)), 42  
[project\\_points\\_to\\_sphere\(\)](#) (in module [clifford.tools.g3c](#)), 42  
[project\\_val\(\)](#) (in module [clifford.tools.g3c](#)), 48  
[pseudoScalar\(\)](#) ([clifford.Layout](#) property), 14  
[pseudoScalar\(\)](#) ([clifford.MultiVector](#) property), 9

## Q

[quaternion\\_and\\_vector\\_to\\_rotor\(\)](#) (in module [clifford.tools.g3c](#)), 39  
[quaternion\\_to\\_matrix\(\)](#) (in module [clifford.tools.g3](#)), 31  
[quaternion\\_to\\_rotor\(\)](#) (in module [clifford.tools.g3](#)), 31

## R

[radius \(clifford.tools.classify.Round attribute\)](#), 58  
[radius\(\)](#) ([clifford.cga.Round](#) property), 24  
[random\\_bivector\(\)](#) (in module [clifford.tools.g3c](#)), 34  
[random\\_circle\(\)](#) (in module [clifford.tools.g3c](#)), 35  
[random\\_circle\\_at\\_origin\(\)](#) (in module [clifford.tools.g3c](#)), 35  
[random\\_conformal\\_point\(\)](#) (in module [clifford.tools.g3c](#)), 37  
[random\\_euc\\_mv\(\)](#) (in module [clifford.tools.g3](#)), 32  
[random\\_line\(\)](#) (in module [clifford.tools.g3c](#)), 35  
[random\\_line\\_at\\_origin\(\)](#) (in module [clifford.tools.g3c](#)), 35  
[random\\_plane\(\)](#) (in module [clifford.tools.g3c](#)), 36  
[random\\_plane\\_at\\_origin\(\)](#) (in module [clifford.tools.g3c](#)), 36  
[random\\_point\\_pair\(\)](#) (in module [clifford.tools.g3c](#)), 35  
[random\\_point\\_pair\\_at\\_origin\(\)](#) (in module [clifford.tools.g3c](#)), 35

[random\\_rotation\\_rotor\(\)](#) (in module [clifford.tools.g3](#)), 32  
[random\\_rotation\\_translation\\_rotor\(\)](#) (in module [clifford.tools.g3c](#)), 37  
[random\\_sphere\(\)](#) (in module [clifford.tools.g3c](#)), 36  
[random\\_sphere\\_at\\_origin\(\)](#) (in module [clifford.tools.g3c](#)), 36  
[random\\_translation\\_rotor\(\)](#) (in module [clifford.tools.g3c](#)), 36  
[random\\_unit\\_vector\(\)](#) (in module [clifford.tools.g3](#)), 32  
[randomIntMV\(\)](#) ([clifford.Layout](#) method), 14  
[randomMV\(\)](#) ([clifford.Layout](#) method), 14  
[randomMV\(\)](#) ([clifford.MultiVector](#) method), 10  
[randomMV\(\)](#) (in module [clifford](#)), 18  
[randomRotor\(\)](#) ([clifford.Layout](#) method), 14  
[randomV\(\)](#) ([clifford.Layout](#) method), 14  
[randomV\(\)](#) ([clifford.MultiVector](#) method), 10  
[right\\_complement\(\)](#) ([clifford.MultiVector](#) method), 8  
[right\\_complement\\_func](#) ([clifford.Layout](#) attribute), 13  
[rightInv\(\)](#) ([clifford.MultiVector](#) method), 9  
[Rotation \(class in clifford.cga\)](#), 25  
[rotation\(\)](#) ([clifford.cga.CGA](#) method), 20  
[rotation\\_matrix\\_align\\_vecs\(\)](#) (in module [clifford.tools.g3](#)), 33  
[rotation\\_matrix\\_to\\_quaternion\(\)](#) (in module [clifford.tools.g3](#)), 31  
[rotor\\_align\\_vecs\(\)](#) (in module [clifford.tools.g3](#)), 33  
[rotor\\_between\\_lines\(\)](#) (in module [clifford.tools.g3c](#)), 55  
[rotor\\_between\\_objects\(\)](#) (in module [clifford.tools.g3c](#)), 54  
[rotor\\_between\\_planes\(\)](#) (in module [clifford.tools.g3c](#)), 55  
[rotor\\_decomp\(\)](#) (in module [clifford.tools](#)), 62  
[rotor\\_mask\(\)](#) ([clifford.Layout](#) property), 13  
[rotor\\_to\\_quaternion\(\)](#) (in module [clifford.tools.g3](#)), 31  
[rotor\\_vector\\_to\\_vector\(\)](#) (in module [clifford.tools.g3](#)), 33  
[Round \(class in clifford.cga\)](#), 22  
[Round \(class in clifford.tools.classify\)](#), 58  
[round\(\)](#) ([clifford.cga.CGA](#) method), 20  
[round\(\)](#) ([clifford.MultiVector](#) method), 8

## S

[scalar\(\)](#) ([clifford.Layout](#) property), 14  
[scale\\_TR\\_translation\(\)](#) (in module [clifford.tools.g3c](#)), 49  
[sig \(clifford.Layout attribute\)](#), 12  
[Sphere \(class in clifford.tools.classify\)](#), 59

`sphere_behind_plane()` (in module `clifford.tools.g3c`), 43  
`sphere_beyond_plane()` (in module `clifford.tools.g3c`), 43  
`square_roots_of_rotor()` (in module `clifford.tools.g3c`), 53  
`standard_line_at_origin()` (in module `clifford.tools.g3c`), 35  
`standard_point_pair_at_origin()` (in module `clifford.tools.g3c`), 35  
`straight_up()` (`clifford.cga.CGA` method), 21

## T

`Tangent` (class in `clifford.tools.classify`), 59  
`Transformation` (in module `clifford.transformations`), 64  
`Translation` (class in `clifford.cga`), 27  
`translation()` (`clifford.cga.CGA` method), 21  
`Transversion` (class in `clifford.cga`), 28  
`transversion()` (`clifford.cga.CGA` method), 21

## U

`ugly()` (in module `clifford`), 16  
`up()` (`clifford.ConformalLayout` method), 15

## V

`val_annihilate_k()` (in module `clifford.tools.g3c`), 52  
`val_apply_rotor()` (in module `clifford.tools.g3c`), 45  
`val_apply_rotor_inv()` (in module `clifford.tools.g3c`), 46  
`val_average_objects()` (in module `clifford.tools.g3c`), 54  
`val_average_objects_with_weights()` (in module `clifford.tools.g3c`), 53  
`val_convert_2D_point_to_conformal()` (in module `clifford.tools.g3c`), 40  
`val_convert_2D_polar_line_to_conformal_line()` (in module `clifford.tools.g3c`), 40  
`val_distance_point_to_line()` (in module `clifford.tools.g3c`), 41  
`val_down()` (in module `clifford.tools.g3c`), 47  
`val_fit_circle()` (in module `clifford.tools.g3c.object_fitting`), 56  
`val_fit_line()` (in module `clifford.tools.g3c.object_fitting`), 56  
`val_fit_plane()` (in module `clifford.tools.g3c.object_fitting`), 57  
`val_fit_sphere()` (in module `clifford.tools.g3c.object_fitting`), 56  
`val_get_line_intersection()` (in module `clifford.tools.g3c`), 42

`val_get_line_reflection_matrix()` (in module `clifford.tools.g3c`), 48  
`val_homo()` (in module `clifford.tools.g3c`), 47  
`val_intersect_line_and_plane_to_point()` (in module `clifford.tools.g3c`), 39  
`val_midpoint_between_lines()` (in module `clifford.tools.g3c`), 41  
`val_midpoint_of_line_cluster()` (in module `clifford.tools.g3c`), 41  
`val_midpoint_of_line_cluster_grad()` (in module `clifford.tools.g3c`), 41  
`val_norm()` (in module `clifford.tools.g3c`), 46  
`val_normalInv()` (in module `clifford.tools.g3c`), 47  
`val_normalise_n_minus_1()` (in module `clifford.tools.g3c`), 45  
`val_normalised()` (in module `clifford.tools.g3c`), 46  
`val_point_pair_to_end_points()` (in module `clifford.tools.g3c`), 39  
`val_rotor_between_lines()` (in module `clifford.tools.g3c`), 55  
`val_rotor_between_objects_explicit()` (in module `clifford.tools.g3c`), 54  
`val_rotor_between_objects_root()` (in module `clifford.tools.g3c`), 54  
`val_rotor_rotor_between_planes()` (in module `clifford.tools.g3c`), 55  
`val_truncated_get_line_reflection_matrix()` (in module `clifford.tools.g3c`), 48  
`val_unsign_sphere()` (in module `clifford.tools.g3c`), 49  
`val_up()` (in module `clifford.tools.g3c`), 47  
`vee()` (`clifford.MultiVector` method), 7  
`vee_func` (`clifford.Layout` attribute), 13

## X

`x()` (`clifford.MultiVector` method), 10